

SMOOTH APPROXIMATION OF THE MODIFIED CONICAL KÄHLER-RICCI FLOW

RYOSUKE TAKAHASHI

ABSTRACT. We introduce the conical Kähler-Ricci flow modified by a holomorphic vector field. We construct a long-time solution of the modified conical Kähler-Ricci flow as the limit of a sequence of smooth Kähler-Ricci flows.

1. INTRODUCTION

Let M be an n -dimensional Fano manifold with a Kähler metric $\omega_0 \in 2\pi c_1(M)$. A Kähler metric $\omega \in 2\pi c_1(M)$ is called *Kähler-Einstein* if it satisfies $\text{Ric}(\omega) = \omega$. For a long while, it was conjectured that the existence of Kähler-Einstein metrics is equivalent to some algebro-geometric stability in the sense of Geometric Invariant Theory (Yau-Donaldson-Tian conjecture), which was recently solved by Chen-Donaldson-Sun [CDS15] and Tian [Tia15]. Their strategy was to study the existence problem of *smooth* Kähler-Einstein metrics on M by deforming the cone angle, i.e., study the Gromov-Hausdorff limit of conical Kähler-Einstein metrics with cone angle $2\pi\beta$ ($0 < \beta \leq 1$) along a smooth divisor $D \in |-K_M|$:

$$\text{Ric}(\omega) = \beta\omega + (1 - \beta)[D]$$

when β goes to 1, where $[D]$ is the current of integration along D . Although YDT conjecture has been completely settled, the existence problem of conical Kähler-Einstein metrics itself is also an interesting problem and studied extensively by many experts (cf. [LS14], [SW16]).

Now we consider more general settings: we allow $D \in |-\lambda K_M|$ ($\lambda \in \mathbb{R}_+$) to be an \mathbb{R} -effective divisor with simple normal crossing support and write

$$D = \sum_{i=1}^d \tau_i D_i$$

where $\tau_i > 0$ and D_i are smooth components. We say that a Kähler current $\omega \in 2\pi c_1(M)$ is a *conical Kähler metric* along $(1 - \beta)D$ ($0 < \beta \leq 1$) if ω is smooth Kähler on $M \setminus D$, and asymptotically equivalent to the model conical Kähler metric near D : more precisely, near each point $p \in \text{Supp}(D)$ where $\text{Supp}(D)$ is cut out by the equation $\{z_1 \cdots z_r = 0\}$ ($r \leq d$) for some local holomorphic coordinates (z^i) , ω satisfies

$$C^{-1}\omega_{\text{model}} \leq \omega \leq C\omega_{\text{model}}$$

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for some constant $C > 0$, where

$$\omega_{\text{model}} := \sqrt{-1} \sum_{i=1}^r |z^i|^{2(\beta-1)\tau_i} dz^i \wedge d\bar{z}^i + \sqrt{-1} \sum_{i=r+1}^n dz^i \wedge d\bar{z}^i$$

is the model conical Kähler metric with cone angles $2\pi(1 - (1 - \beta)\tau_i)$ along $\{z^i = 0\}$. Let X be a holomorphic vector field on M whose imaginary part $\text{Im}(X)$ generates a torus action on the line bundles $\mathcal{O}_M(D_i)$. Let H_i be $\text{Im}(X)$ -invariant hermitian metrics on $\mathcal{O}_M(D_i)$ such that the curvature of the induced hermitian metric $H_D := \otimes_{i=1}^d H_i^{\tau_i}$ is $\lambda\omega_0$. Let s_i be the defining sections of $\mathcal{O}_M(D_i)$ associated to D_i , and set $s_D := \otimes_{i=1}^d s_i^{\tau_i}$. We define a Kähler current ω^* as

$$\omega^* := \omega_0 + k \sum_{i=1}^d \sqrt{-1} \partial \bar{\partial} |s_i|_{H_i}^{2(1-(1-\beta)\tau_i)}$$

for sufficiently small constant $k > 0$. Then ω^* is a conical Kähler metric along $(1 - \beta)D$. According to [DGSW13], we say that a conical Kähler metric $\omega \in c_1(M)$ is a *conical Kähler-Ricci soliton* if it satisfies

$$(1.1) \quad \text{Ric}(\omega) = \gamma\omega + (1 - \beta)[D] + L_X\omega$$

in the sense of distributions on M , and

$$\text{Ric}(\omega) = \gamma\omega + L_X\omega$$

in the classical sense on $M \setminus D$, where $\gamma = \gamma(\lambda, \beta) := 1 - \lambda(1 - \beta) \geq 0$ and $L_X\omega$ is defined so that

$$\int_M L_X\omega \wedge \zeta = - \int_M \omega \wedge L_X\zeta$$

for any smooth $(n - 1, n - 1)$ -form ζ on M . The notion of conical Kähler-Ricci solitons is a generalization of classical Kähler-Ricci solitons (cf. [TZ00], [TZ02]) for the conical settings, and their examples in toric Fano manifolds are studied in [DGSW13] and [WZZ16].

In this paper, we introduce the following *modified conical Kähler-Ricci flow* (MCKRF):

$$(1.2) \quad \begin{cases} \frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) + \gamma\omega + (1 - \beta)[D] + L_X\omega \\ \omega|_{t=0} = \omega^*. \end{cases}$$

Then conical Kähler-Ricci solitons with respect to X can be viewed as the stationary points of MCKRF. We say that $\omega = \omega(t)$ ($t \in [0, \infty)$) is a long-time solution of the above MCKRF if $\omega(t)$ is a conical Kähler metric along $(1 - \beta)D$ for each t which satisfies the equation (1.2) in the sense of distributions on $M \times [0, \infty)$ and can be simplified to the classical modified Kähler-Ricci flow

$$\frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) + \gamma\omega + L_X\omega$$

on $(M \setminus D) \times [0, \infty)$. If a long-time solution of the flow (1.2) converges to some Kähler current, it should be a conical Kähler-Ricci soliton with respect to X . Thus the flow (1.2) provides a new standard method for studying the equation (1.1). In the case when $X \equiv 0$, Chen-Wang [CW15]¹ established the short-time existence of the flow (1.2). Then Liu-Zhang [LZ17] and Wang [Wan16] showed the long-time

¹More precisely, they dealt with the “strong” conical Kähler-Ricci flow (with some Hölder continuity assumptions for potential functions).

existence independently. On the other hand, in the general case, it seems that the flow (1.2) is considered only for $D = 0$ (cf. [TZ07], [PSSW11]).

Following the idea of [LZ17] and [Wan16], we will construct a long-time solution of (1.2) as the limit of a sequence of smooth Kähler-Ricci flows φ_ϵ , where φ_ϵ ($\epsilon > 0$) is a solution of the *modified twisted Kähler-Ricci flow* (MTKRF) defined in Section 2. Then we show the following:

Theorem 1.1. *Assume that $|X(\log |s_D|_{H_D}^2)| < C$ on $M \setminus D$ for some constant $C > 0$. Let $\omega_{\varphi_\epsilon}$ be a long-time solution of the modified twisted Kähler-Ricci flow (2.4). Then, by passing to a subsequence $\{\epsilon_i\}$ satisfying $\epsilon_i \rightarrow 0$ as $i \rightarrow \infty$, the Kähler metric $\omega_{\varphi_{\epsilon_i}}$ converges to a solution of the modified conical Kähler-Ricci flow:*

$$\begin{cases} \frac{\partial \omega_\varphi}{\partial t} = -\text{Ric}(\omega_\varphi) + \gamma \omega_\varphi + (1 - \beta)[D] + L_X \omega_\varphi \\ \omega_\varphi|_{t=0} = \omega^* \end{cases}$$

as $i \rightarrow \infty$, where $\omega_\varphi := \omega^* + \sqrt{-1} \partial \bar{\partial} \varphi$, and for any $t \in [0, \infty)$, the potential function φ is Hölder continuous with respect to ω_0 . This convergence holds in the sense of distributions on $M \times [0, \infty)$, and in the C_{loc}^∞ -topology on $(M \setminus D) \times [0, \infty)$. In particular, there exists a long-time solution of the modified conical Kähler-Ricci flow.

Remark 1.1. (1) The assumption $|X(\log |s_D|_{H_D}^2)| < C$ is a necessary condition for the existence of a conical Kähler-Ricci soliton with respect to X . In particular, this condition implies that X is tangent to $\text{Supp}(D)$ (cf. [JLZ16, Remark 4.2]). This assumption is used only for the uniform Laplacian estimate of MTKRF (cf. Proposition 3.2).
 (2) We also note that when D is smooth and $\lambda \geq 1$, such a vector field X automatically becomes trivial (cf. [SW16, Theorem 2.1]). This is a reason why we allow D to have simple normal crossing support.

An advantage of our approach is that we do not rely on the linear theory for conical Laplacians established by Donaldson [Don12] and Chen-Wang [CW15]. At the same time, we should point out that Theorem 1.1 provides us not only the long-time existence of solutions, but also “the regularization method” to study the flow. The author expects that the conical Kähler-Ricci flow (and its regularization) method also works for the existence problem of conical Kähler-Ricci solitons. The arguments in this paper run closely in parallel to those of [LZ17] except some changes due to the modification X . Nevertheless, we will try to make the arguments reasonably self-contained for readers’ convenience.

The paper is organized as follows. We first review the regularization method and reduction to the Monge-Ampère flow in Section 2. Then we consider the uniform Laplacian estimate for MTKRF in Section 3. Finally, we establish the C_{loc}^∞ -estimate of MTKRF and give the proof of Theorem 1.1 in Section 4.

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2. REGULARIZATION AND REDUCTION TO THE MONGE-AMPÈRE FLOW

Let $\epsilon > 0$ be a small constant. As in [GP16, Section 3.1], We define the function

$$(2.1) \quad \chi_i(\epsilon^2 + u) := \frac{1}{1 - (1 - \beta)\tau_i} \int_0^u \frac{(\epsilon^2 + r)^{1-(1-\beta)\tau_i} - \epsilon^{2(1-(1-\beta)\tau_i)}}{r} dr$$

for $i = 1, \dots, d$ and $u \geq 0$. Then we see that the function $\chi_i(\epsilon^2 + u)$ is smooth for each ϵ , and there exists uniform constants (independent of ϵ) $C > 0$ and $\nu > 0$ such that for all i , we have

$$(2.2) \quad 0 \leq \chi_i(\epsilon^2 + u) < C$$

provided that u belongs to a bounded interval, and

$$(2.3) \quad \omega_\epsilon \geq \nu \omega_0.$$

We also have the convergence

$$\chi_i(\epsilon^2 + |s_i|_{H_i}^2) \xrightarrow{\epsilon \rightarrow 0} |s_i|_{H_i}^{2(1-(1-\beta)\tau_i)}$$

in the C_{loc}^∞ -topology on $M \setminus D_i$. Set $\chi := \sum_{i=1}^d \chi_i(\epsilon^2 + |s_i|_{H_i}^2)$ and $\omega_\epsilon := \omega_0 + \sqrt{-1} \partial \bar{\partial} k \chi$. Then we have

$$\omega_\epsilon \xrightarrow{\epsilon \rightarrow 0} \omega^*$$

in the sense of distributions on M , and in the C_{loc}^∞ -topology on $M \setminus D$. Meanwhile, since $[D] = \lambda \omega_0 + \sum_{i=1}^d \sqrt{-1} \tau_i \partial \bar{\partial} \log |s_i|_{H_i}^2$ by the Poincarè-Lelong formula, we observe that

$$\eta_\epsilon := \lambda \omega_0 + \sum_{i=1}^d \sqrt{-1} \tau_i \partial \bar{\partial} \log(|s_i|_{H_i}^2 + \epsilon^2) \xrightarrow{\epsilon \rightarrow 0} [D],$$

again, this convergence holds in the sense of distributions on M , and in the C_{loc}^∞ -topology on $M \setminus D$. Now We define the *modified twisted Kähler-Ricci flow* (MTKRF) with the twisted form η_ϵ :

$$(2.4) \quad \begin{cases} \frac{\partial \omega_{\varphi_\epsilon}}{\partial t} = -\text{Ric}(\omega_{\varphi_\epsilon}) + \gamma \omega_{\varphi_\epsilon} + (1 - \beta) \eta_\epsilon + L_X \omega_{\varphi_\epsilon} \\ \omega_{\varphi_\epsilon}|_{t=0} = \omega_\epsilon, \end{cases}$$

where $\omega_{\varphi_\epsilon} := \omega_\epsilon + \sqrt{-1} \partial \bar{\partial} \varphi_\epsilon$. For an $\text{Im}(X)$ -invariant Kähler metric $\omega \in 2\pi c_1(M)$, we also define an \mathbb{R} -valued function $\theta_X(\omega)$ by

$$(2.5) \quad \begin{cases} i_X \omega = \sqrt{-1} \bar{\partial} \theta_X(\omega) \\ \int_M e^{\theta_X(\omega)} \omega^n = [\omega_0]^n. \end{cases}$$

In particular, we set $\theta_X := \theta_X(\omega_0)$. Then, from [TZ02, Proposition 1.1] and [Zhu00, Corollary 5.3] (or [BN14, Section 2.3]), we have the following:

Proposition 2.1. *Let ϕ be a real-valued smooth function such that $\text{Im}(X)(\phi) = 0$ and $\omega_\phi := \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi \geq 0$. Then we have*

- (1) $\theta_X(\omega_\phi) = \theta_X + X(\phi)$.
- (2) $\sup_M |X(\phi)| < C$ for some constant C which depends only on ω_0 and X .

Since MTKRF preserves the initial Kähler class $[\omega_0]$, we can reduce MTKRF to the Monge-Ampère flow:

$$(2.6) \quad \begin{cases} \frac{\partial \varphi_\epsilon}{\partial t} = \log \frac{\omega_{\varphi_\epsilon}^n}{\omega_0^n} + F_0 + \gamma(k\chi + \varphi_\epsilon) + \log(\prod_{i=1}^d (\epsilon^2 + |s_i|_{H_i}^2))^{(1-\beta)\tau_i} + \theta_X(\omega_{\varphi_\epsilon}) \\ \varphi_\epsilon|_{t=0} = c_{\epsilon 0}. \end{cases}$$

where $c_{\epsilon 0}$ is a real constant such that $c_{\epsilon 0} \xrightarrow{\epsilon \rightarrow 0} c_0$ and F_0 is the Ricci potential with respect to ω_0 :

$$(2.7) \quad \begin{cases} -\text{Ric}(\omega_0) + \omega_0 = \sqrt{-1}\partial\bar{\partial}F_0 \\ \int_X e^{-F_0} \omega_0^n = [\omega_0]^n. \end{cases}$$

We often use the twisted Ricci potential F_ϵ defined by

$$F_\epsilon := F_0 + \log \left(\frac{\omega_\epsilon^n}{\omega_0^n} \cdot \prod_{i=1}^d (\epsilon^2 + |s_i|_{H_i}^2)^{(1-\beta)\tau_i} \right).$$

Remark 2.1. According to [CGP13], we see that F_ϵ is uniformly bounded.

Then the flow (2.6) can be written as

$$\begin{cases} \frac{\partial \varphi_\epsilon}{\partial t} = \log \frac{\omega_{\varphi_\epsilon}^n}{\omega_\epsilon^n} + F_\epsilon + \gamma(k\chi + \varphi_\epsilon) + \theta_X(\omega_{\varphi_\epsilon}) \\ \varphi_\epsilon|_{t=0} = c_{\epsilon 0}. \end{cases}$$

3. C^0 -ESTIMATE, VOLUME RATIO ESTIMATE AND UNIFORM LAPLACIAN ESTIMATE

In this section, we establish the uniform Laplacian estimate of MTKRF. First, we show the volume ratio estimate and C^0 -estimate:

Proposition 3.1. *Let φ_ϵ be the solution of (2.6). Then there exists a uniform constant C (independent of ϵ and t) such that*

$$\sup_{M \times [0, T]} |\varphi_\epsilon| \leq C^{\gamma T},$$

$$\sup_{M \times [0, T]} |\dot{\varphi}_\epsilon| \leq C e^{\gamma T}.$$

Proof. Differentiating the equation (2.6) in t , we have

$$\frac{d\dot{\varphi}_\epsilon}{dt} = (\Delta_{\omega_{\varphi_\epsilon}} + X)\dot{\varphi}_\epsilon + \gamma\dot{\varphi}_\epsilon.$$

By the maximum principle, we have

$$|\dot{\varphi}_\epsilon(t)| \leq |\dot{\varphi}(0)|e^{\gamma t},$$

where $\dot{\varphi}(0) = F_\epsilon + \gamma(k\chi + c_{\epsilon 0}) + \theta_X + X(k\chi)$. Thus, by (2.2), Proposition 2.1 and Remark 2.1, we know that $|\dot{\varphi}(0)| \leq C$ for some uniform constant C . Then we have

$$|\dot{\varphi}_\epsilon(t)| \leq C e^{\gamma t}.$$

Integrating with respect to t , we get

$$|\varphi_\epsilon(t)| \leq C e^{\gamma t}$$

as desired. \square

As in the arguments in [LZ17, Proposition 3.1] and [JLZ16, Theorem 4.3], we can show the uniform Laplacian estimate for MTKRF:

Proposition 3.2. *Let φ_ϵ be a solution of (2.6). Assume that there exists a uniform constant $C > 0$ such that*

- (1) $\sup_{M \times [0, T]} |\varphi_\epsilon| < C$,
- (2) $\sup_{M \times [0, T]} |\dot{\varphi}_\epsilon| < C$.

Then there exists a uniform constant $A = A(\lambda, \{\tau_i\}, \beta, \omega_0, X, C)$ such that

$$(3.1) \quad A^{-1}\omega_\epsilon \leq \omega_{\varphi_\epsilon} \leq A\omega_\epsilon.$$

Proof. We choose local normal coordinates (z^i) with respect to ω_ϵ where $\omega_{\varphi_\epsilon}$ is diagonal, and reduce to local computation. Then we observe that

$$\begin{aligned} \left(\frac{d}{dt} - \Delta_{\omega_{\varphi_\epsilon}} \right) \log \operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon} &= \frac{1}{\operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \left(\Delta_{\omega_\epsilon} \left(\dot{\varphi}_\epsilon - \log \frac{\omega_{\varphi_\epsilon}^n}{\omega_\epsilon^n} \right) + R_{\omega_\epsilon} \right) \\ &\quad - \frac{1}{\operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} (g_{\varphi_\epsilon}^{p\bar{q}} g_{\varphi_\epsilon j\bar{m}} R_{\omega_\epsilon p\bar{q}}^{\bar{m}j}) \\ &\quad + \left\{ \frac{g_{\varphi_\epsilon}^{\delta\bar{k}} \partial_\delta \operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon} \partial_{\bar{k}} \operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}}{(\operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon})^2} - \frac{g_\epsilon^{\gamma\bar{s}} \varphi_{\epsilon\gamma}^t p \varphi_{\epsilon\bar{s}t}^p}{\operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \right\}. \end{aligned}$$

The computation in [Tos15, Theorem 3.9] implies that

$$\frac{g_{\varphi_\epsilon}^{\delta\bar{k}} \partial_\delta \operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon} \partial_{\bar{k}} \operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}}{(\operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon})^2} - \frac{g_\epsilon^{\gamma\bar{s}} \varphi_{\epsilon\gamma}^t p \varphi_{\epsilon\bar{s}t}^p}{\operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \leq 0.$$

Since

$$\begin{aligned} g_{\varphi_\epsilon}^{p\bar{q}} g_{\varphi_\epsilon j\bar{m}} R_{\omega_\epsilon p\bar{q}}^{\bar{m}j} &= \frac{1 + \varphi_{\epsilon i\bar{i}}}{1 + \varphi_{\epsilon j\bar{j}}} R_{\omega_\epsilon j\bar{j}}^{i\bar{i}}, \\ n &= \operatorname{tr}_{\omega_\epsilon} \omega_0 + k \operatorname{tr}_{\omega_\epsilon} (\sqrt{-1} \partial \bar{\partial} \chi) \geq k \Delta_{\omega_\epsilon} \chi, \\ \frac{\Delta_{\omega_\epsilon} \varphi_\epsilon}{\operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} &= \frac{\sum_i \varphi_{\epsilon i\bar{i}}}{\sum_i (1 + \varphi_{\epsilon i\bar{i}})} \leq 1, \end{aligned}$$

we have

$$\begin{aligned} \left(\frac{d}{dt} - \Delta_{\omega_{\varphi_\epsilon}} \right) \log \operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon} &\leq -\frac{1}{\operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \sum_{i,j} \frac{1 + \varphi_{\epsilon i\bar{i}}}{1 + \varphi_{\epsilon j\bar{j}}} R_{\omega_\epsilon j\bar{j}}^{i\bar{i}} \\ &\quad + \frac{1}{\operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \Delta_{\omega_\epsilon} (F_\epsilon + \gamma(k\chi + \varphi_\epsilon) + \theta_X(\omega_{\varphi_\epsilon})) + R_{\omega_\epsilon} \\ &\leq -\frac{1}{\operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \sum_{i \leq j} \left(\frac{1 + \varphi_{\epsilon i\bar{i}}}{1 + \varphi_{\epsilon j\bar{j}}} + \frac{1 + \varphi_{\epsilon j\bar{j}}}{1 + \varphi_{\epsilon i\bar{i}}} - 2 \right) R_{\omega_\epsilon j\bar{j}}^{i\bar{i}} \\ &\quad + \frac{1}{\operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} (\Delta_{\omega_\epsilon} F_\epsilon) + \frac{\gamma n}{\operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} + \gamma + \frac{1}{\operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \Delta_{\omega_\epsilon} \theta_X(\omega_{\varphi_\epsilon}). \end{aligned}$$

Let C_1 be a uniform constant such that

$$\sqrt{-1} \partial \bar{\partial} F_0 \geq -C_1 \omega_0.$$

Then, by (2.3), we have

$$0 \leq \operatorname{tr}_{\omega_\epsilon} (\sqrt{-1} \partial \bar{\partial} F_0 + C_1 \omega_0) \leq \nu^{-1} \operatorname{tr}_{\omega_0} (\sqrt{-1} \partial \bar{\partial} F_0 + C_1 \omega_0) = \nu^{-1} (C_1 n + \Delta_{\omega_0} F_0).$$

Hence we have the uniform bound of $\Delta_{\omega_\epsilon} F_0$:

$$-C_1\nu^{-1} \leq -C_1\mathrm{tr}_{\omega_\epsilon}\omega_0 \leq \Delta_{\omega_\epsilon}F_0 \leq \nu^{-1}(C_1n + \Delta_{\omega_0}F_0).$$

Now we recall the arguments in [GP16, Section 2, Section 3, Section 4]. We set

$$\chi_\rho(\epsilon^2 + u) = \frac{1}{\rho} \int_0^u \frac{(\epsilon^2 + r)^\rho - \epsilon^{2\rho}}{r} dr$$

and define the “auxiliary function” $\Psi_{\epsilon,\rho}$ by

$$\Psi_{\epsilon,\rho} := \tilde{C} \sum_{i=1}^d \chi_\rho(\epsilon^2 + |s_i|_{H_i}^2),$$

where $\tilde{C} > 0$ and $\rho > 0$ are constants. Then the function $\Psi_{\epsilon,\rho}$ is uniformly bounded. After taking suitable uniform constants \tilde{C} , ρ and C_2 , we have

$$\begin{aligned} & - \sum_{i \geq j} \left(\frac{1 + \varphi_{\epsilon i \bar{i}}}{1 + \varphi_{\epsilon j \bar{j}}} + \frac{1 + \varphi_{\epsilon j \bar{j}}}{1 + \varphi_{\epsilon i \bar{i}}} - 2 \right) R_{\omega_\epsilon j \bar{j}}^{i \bar{i}} - \mathrm{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon} \Delta_{\omega_{\varphi_\epsilon}} \Psi_{\epsilon,\rho} + \Delta_{\omega_\epsilon} F_\epsilon \\ & \leq C_2 \sum_{i \leq j} \left(\frac{1 + \varphi_{\epsilon i \bar{i}}}{1 + \varphi_{\epsilon j \bar{j}}} + \frac{1 + \varphi_{\epsilon j \bar{j}}}{1 + \varphi_{\epsilon i \bar{i}}} \right) + C_2 \mathrm{tr}_{\omega_{\varphi_\epsilon}} \omega_\epsilon \cdot \mathrm{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon} + \Delta_{\omega_\epsilon} F_0 + C_2. \end{aligned}$$

Combining with the Cauchy-Schwartz inequality $n \leq \mathrm{tr}_{\omega_{\varphi_\epsilon}} \omega_\epsilon \cdot \mathrm{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}$, we get

$$\begin{aligned} \left(\frac{d}{dt} - \Delta_{\omega_{\varphi_\epsilon}} \right) (\log \mathrm{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon} + \Psi_{\epsilon,\rho}) & \leq \frac{C_2}{\mathrm{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \sum_{i \leq j} \left(\frac{1 + \varphi_{\epsilon i \bar{i}}}{1 + \varphi_{\epsilon j \bar{j}}} + \frac{1 + \varphi_{\epsilon j \bar{j}}}{1 + \varphi_{\epsilon i \bar{i}}} \right) \\ & \quad + \frac{C_3}{\mathrm{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} + C_2 \mathrm{tr}_{\omega_{\varphi_\epsilon}} \omega_\epsilon + \frac{1}{\mathrm{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \Delta_{\omega_\epsilon} \theta_X(\omega_{\varphi_\epsilon}) + C_4 \\ & \leq \frac{C_2}{\mathrm{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \left\{ \left(\sum_i \frac{1}{1 + \varphi_{\epsilon i \bar{i}}} \right) \left(\sum_j (1 + \varphi_{\epsilon j \bar{j}}) \right) + n \right\} \\ & \quad + \frac{C_3}{\mathrm{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} + C_2 \mathrm{tr}_{\omega_{\varphi_\epsilon}} \omega_\epsilon + \frac{1}{\mathrm{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \Delta_{\omega_\epsilon} \theta_X(\omega_{\varphi_\epsilon}) + C_4 \\ & \leq C_5 \mathrm{tr}_{\omega_{\varphi_\epsilon}} \omega_\epsilon + \frac{1}{\mathrm{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \Delta_{\omega_\epsilon} \theta_X(\omega_{\varphi_\epsilon}) + C_4. \end{aligned}$$

Since $\max_{i=1,\dots,n} \{\sup_M X^i_{,i}, 0\} \leq C_6$ is uniformly bounded² (cf. [JLZ16, Lemma A.2]), we get

$$\begin{aligned}
\Delta_{\omega_\epsilon} \theta_X(\omega_{\varphi_\epsilon}) &= \sum_i \theta_X(\omega_{\varphi_\epsilon})_{i\bar{i}} \\
&= \sum_i (X^j g_{\varphi_\epsilon j \bar{i}})_{i\bar{i}} \\
&= \sum_i (X^j_{,i} g_{\varphi_\epsilon j \bar{i}} + X^j g_{\varphi_\epsilon j \bar{i}, i}) \\
&= \sum_i (X^j_{,i} g_{\varphi_\epsilon j \bar{i}} + X^j g_{\varphi_\epsilon i \bar{i}, j}) \\
&= \sum_i X^i_{,i} (1 + \varphi_{\epsilon i \bar{i}}) + \sum_i X^j \varphi_{\epsilon i \bar{i} j} \\
&\leq C_6 \operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon} + \sum_i X^j \varphi_{\epsilon i \bar{i} j}.
\end{aligned}$$

On the other hand, from the assumption (2), we know that

$$\left(\frac{d}{dt} - \Delta_{\omega_{\varphi_\epsilon}} \right) \varphi_\epsilon = \dot{\varphi}_\epsilon - \operatorname{tr}_{\omega_{\varphi_\epsilon}} (\omega_{\varphi_\epsilon} - \omega_\epsilon) = \dot{\varphi}_\epsilon - n + \operatorname{tr}_{\omega_{\varphi_\epsilon} \omega_\epsilon} \geq \operatorname{tr}_{\omega_{\varphi_\epsilon} \omega_\epsilon} - (C + n).$$

Thus, if we set $B := C_5 + 1$, we have

$$\left(\frac{d}{dt} - \Delta_{\omega_{\varphi_\epsilon}} \right) (\log \operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon} + \Psi_{\epsilon, \rho} - B\varphi_\epsilon) \leq -\operatorname{tr}_{\omega_{\varphi_\epsilon} \omega_\epsilon} + \frac{1}{\operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \sum_i X^j \varphi_{\epsilon j \bar{j} i} + C_7.$$

We assume that the function $\log \operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon} + \Psi_{\epsilon, \rho} - B\varphi_\epsilon$ takes its maximum at $(x_0, t_0) \in M \times [0, T]$. If $t_0 = 0$, we have $\log \operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon} + \Psi_{\epsilon, \rho} - B\varphi_\epsilon = \log n + \Psi_{\epsilon, \rho} - Bc_{\epsilon 0}$, which is uniformly bounded since $\Psi_{\epsilon, \rho}$ and $c_{\epsilon 0}$ is. Now we assume that $t_0 > 0$. Then, by the maximum principle, we have

$$0 \leq -\operatorname{tr}_{\omega_{\varphi_\epsilon} \omega_\epsilon} + \frac{1}{\operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \sum_i X^j \varphi_{\epsilon j \bar{j} i} + C_7$$

at (x_0, t_0) . On the other hand, differentiating the function $\log \operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon} + \Psi_{\epsilon, \rho} - B\varphi_\epsilon$ in z^j implies

$$\frac{\partial}{\partial z^j} (\log \operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon} + \Psi_{\epsilon, \rho} - B\varphi_\epsilon) = \frac{1}{\operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \varphi_{\epsilon i \bar{i} j} + \Psi_{\epsilon, \rho, j} - B\varphi_{\epsilon j}.$$

Hence, at (x_0, t_0) , we have

$$\frac{1}{\operatorname{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \sum_i X^j \varphi_{\epsilon j \bar{j} i} = X(B\varphi_\epsilon - \Psi_{\epsilon, \rho}).$$

According to [GP16, Section 4], we find that there exists a small uniform constant $k' > 0$ such that $\omega_0 + k'\sqrt{-1}\partial\bar{\partial}\Psi_{\epsilon, \rho} \geq 0$. Thus, combining with Proposition 2.1 implies

$$\begin{aligned}
|X(\varphi_\epsilon)| &\leq |X(k\chi + \varphi_\epsilon)| + |X(k\chi)| \leq C_8, \\
|X(\Psi_{\epsilon, \rho})| &\leq C_9.
\end{aligned}$$

Thus we have

$$\operatorname{tr}_{\omega_{\varphi_\epsilon}} \omega_\epsilon \leq C_{10}$$

²We need the assumption $|X(\log |s_D|_{H_D}^2)| < C$ to get this uniform bound.

at (x_0, t_0) . Then we observe that

$$\begin{aligned} \mathrm{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}(x_0, t_0) &\leq \frac{1}{(n-1)!} (\mathrm{tr}_{\omega_{\varphi_\epsilon}} \omega_\epsilon)^{n-1}(x_0, t_0) \cdot \frac{\omega_{\varphi_\epsilon}^n}{\omega_\epsilon^n}(x_0, t_0) \\ &\leq \frac{C_{10}^{n-1}}{(n-1)!} \exp(\dot{\varphi}_\epsilon - F_\epsilon - \gamma(k\chi + \varphi_\epsilon) - \theta_X - X(k\chi + \varphi_\epsilon))(x_0, t_0) \\ &\leq C_{11}. \end{aligned}$$

Since F_ϵ and $\Psi_{\epsilon, \rho}$ are uniformly bounded, we find that

$$\mathrm{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon} \leq C_{12}$$

on M . Hence the flow equation (2.6) and the uniform bound of $\varphi_\epsilon, \dot{\varphi}_\epsilon, F_\epsilon, X(k\chi + \varphi_\epsilon)$ give the desired inequality (3.1) for some uniform constant A . \square

4. C_{loc}^∞ -ESTIMATE AND COMPLETION OF THE PROOF OF THEOREM 1.1

In this section, we establish the C_{loc}^∞ -estimate of MTKRF. Let

$$\phi_\epsilon := \varphi_\epsilon + k\chi.$$

Then we have

$$\omega_{\phi_\epsilon} := \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi = \omega_{\varphi_\epsilon}.$$

In order to simplify the notation, we drop the explicit dependence of ϵ and write ϕ, η , etc. Then the equation of MTKRF can be written as

$$(4.1) \quad \frac{\partial \omega_\phi}{\partial t} = -\mathrm{Ric}(\omega_\phi) + \gamma \omega_\phi + \tilde{\eta} + L_X \omega_\phi,$$

where $\tilde{\eta} := (1 - \beta)\eta \in (1 - \gamma)c_1(M)$, or equivalently,

$$(4.2) \quad \frac{dg_{\phi k \bar{l}}}{dt} = -R_{\phi k \bar{l}} + \gamma g_{\phi k \bar{l}} + \tilde{\eta}_{k \bar{l}} + \nabla_{\phi k} X_{\bar{l}}.$$

Then we can reduce the above equation to the Monge-Ampère flow:

$$(4.3) \quad \frac{\partial \phi}{\partial t} = \log \frac{\omega_\phi^n}{\omega_0^n} + \gamma \phi + F + \theta_X(\omega_\phi),$$

where F is a twisted Ricci potential $\sqrt{-1} \partial \bar{\partial} F = -\mathrm{Ric}(\omega_0) + \gamma \omega_0 + \tilde{\eta}$. Let ∇_ϕ (resp. ∇_0) be the covariant derivative with respect to ω_ϕ (resp. ω_0). We set

$$S := |\nabla_0 g_\phi|_{\omega_\phi}^2 = g_\phi^{i\bar{j}} g_\phi^{k\bar{l}} g_\phi^{p\bar{q}} \nabla_{0i} g_{\phi k \bar{q}} \nabla_{0\bar{j}} g_{\phi p \bar{l}}.$$

If we put

$$\begin{aligned} h^i_k &:= g_0^{i\bar{j}} g_{\phi k \bar{j}}, \\ U_{il}^k &:= (\nabla_{\phi i} h \cdot h^{-1})^k_l, \end{aligned}$$

then we have

$$(4.4) \quad \begin{aligned} U_{il}^k &= \Gamma_{\phi il}^k - \Gamma_{0il}^k, \\ S &= |U|_{\omega_\phi}^2, \end{aligned}$$

where $\Gamma_{\phi il}^k$ (resp. Γ_{0il}^k) is the Christoffel symbol of ω_ϕ (resp. ω_0). The following proposition is an X -analogue of [LZ17, Proposition 3.3].

Proposition 4.1. *Let $p \in M$ and ϕ be a solution of the Monge-Ampère flow (4.3). We assume that there exists a constant $N > 0$ such that*

$$(4.5) \quad N^{-1}\omega_0 \leq \omega_\phi \leq N\omega_0$$

on $B_r(p) \times [0, T]$, where $B_r(p)$ is a geodesic ball of radius $r > 0$ centered at p with respect to ω_0 . Then there exists constants

$$C' = C'(N, \gamma, \omega_0, X, \|\phi(\cdot, 0)\|_{C^3(B_r(p))}, \|\tilde{\eta}\|_{C^1(B_r(p))})$$

and

$$C'' = C''(N, \gamma, \omega_0, X, \|\phi(\cdot, 0)\|_{C^4(B_r(p))}, \|\tilde{\eta}\|_{C^2(B_r(p))})$$

such that

$$S \leq C',$$

$$|\text{Rm}_\phi|_{\omega_\phi}^2 \leq C''$$

on $B_{r/2}(p) \times [0, T]$. Moreover, for any $k \geq 0$ and $0 < \alpha < 1$, there exists constants

$$C_k^i = C_k^i(N, \gamma, \omega_0, X, \|\phi(\cdot, 0)\|_{C^{k+4}(B_r(p))}, \|\phi\|_{C^0(B_r(p) \times [0, T])}, \|\tilde{\eta}\|_{C^{k+2}(B_r(p))}, \|F\|_{C^0(B_r(p))}) \quad (i = 1, 2, 3)$$

such that

$$|D^k \text{Rm}_\phi|_{\omega_\phi}^2 \leq C_k^1,$$

$$\|\dot{\phi}\|_{C^{k+1, \alpha}} \leq C_k^2,$$

$$\|\phi\|_{C^{k+3, \alpha}} \leq C_k^3$$

on $B_{r/2}(p) \times [0, T]$.

Proof. We first establish the local version of Calabi's C^3 -estimate. A direct computation shows that

$$\begin{aligned} \left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) S &= g_\phi^{m\bar{\gamma}} g_{\phi\bar{\mu}\beta} g_\phi^{l\bar{\alpha}} ((g_\phi^{\beta\bar{s}} \nabla_{\phi m} \tilde{\eta}_{\bar{s}l} - \nabla_\phi^{\bar{q}} R_0^\beta{}_{l\bar{q}m}) U_{\bar{\gamma}\bar{\alpha}}^{\bar{\mu}} + U_{ml}^\beta (g_\phi^{\bar{\mu}s} \nabla_{\phi\bar{\gamma}} \tilde{\eta}_{s\bar{\alpha}} - \nabla_\phi^q R_0^{\bar{\mu}}{}_{\bar{\alpha}q\bar{\gamma}})) \\ &\quad - U_{ml}^\beta U_{\bar{\gamma}\bar{\alpha}}^{\bar{\mu}} (\tilde{\eta}_{p\bar{q}} g_\phi^{p\bar{\gamma}} g_\phi^{m\bar{q}} g_{\phi\bar{\mu}\beta} g_\phi^{l\bar{\alpha}} - g_\phi^{m\bar{\gamma}} \tilde{\eta}_{\bar{\mu}\beta} g_\phi^{l\bar{\alpha}} + g_\phi^{m\bar{\gamma}} g_{\phi\bar{\mu}\beta} g_\phi^{p\bar{\alpha}} g_\phi^{l\bar{q}} \tilde{\eta}_{p\bar{q}}) \\ &\quad - \gamma S - |\nabla_\phi U|_{\omega_\phi}^2 - |\bar{\nabla}_\phi U|_{\omega_\phi}^2 \\ &\quad + \underbrace{g_\phi^{m\bar{\gamma}} g_{\phi\bar{\mu}\beta} g_\phi^{l\bar{\alpha}} \cdot \nabla_{\phi m} \nabla_{\phi l} X^\beta \cdot U_{\bar{\gamma}\bar{\alpha}}^{\bar{\mu}}}_{(X;I)} + \underbrace{g_\phi^{m\bar{\gamma}} g_{\phi\bar{\mu}\beta} g_\phi^{l\bar{\alpha}} \cdot \nabla_{\phi\bar{\gamma}} \nabla_{\phi\bar{\alpha}} X^{\bar{\mu}} \cdot U_{ml}^\beta}_{(X;II)} \\ &\quad - \underbrace{g_{\phi\bar{\mu}\beta} g_\phi^{l\bar{\alpha}} \cdot \nabla_\phi^{\bar{\gamma}} X^m \cdot U_{ml}^\beta U_{\bar{\gamma}\bar{\alpha}}^{\bar{\mu}}}_{(X;III)} + \underbrace{g_\phi^{m\bar{\gamma}} g_\phi^{l\bar{\alpha}} \cdot \nabla_{\phi\beta} X_{\bar{\mu}} \cdot U_{ml}^\beta U_{\bar{\gamma}\bar{\alpha}}^{\bar{\mu}}}_{(X;IV)} \\ &\quad - \underbrace{g_\phi^{m\bar{\gamma}} g_{\phi\bar{\mu}\beta} \cdot \nabla_\phi^{\bar{\alpha}} X^l \cdot U_{ml}^\beta U_{\bar{\gamma}\bar{\alpha}}^{\bar{\mu}}}_{(X;V)}, \end{aligned}$$

where (X;I)-(X;V) are additional terms arising from the holomorphic vector field X . Since

$$(4.6) \quad \nabla_{\phi m} \tilde{\eta}_{l\bar{q}} = \nabla_{0m} \tilde{\eta}_{l\bar{q}} - U_{ml}^s \tilde{\eta}_{s\bar{q}},$$

$$(4.7) \quad \nabla_{\phi p} R_0^\beta{}_{l\bar{q}m} = \nabla_{0p} R_0^\beta{}_{l\bar{q}m} + U_{ps}^\beta R_0^s{}_{l\bar{q}m} - U_{pl}^s R_0^\beta{}_{s\bar{q}m} - U_{pm}^s R_0^\beta{}_{l\bar{q}s}.$$

we have

$$g_\phi^{m\bar{\gamma}} g_{\phi\bar{\mu}\beta} g_\phi^{l\bar{\alpha}} ((g_\phi^{\beta\bar{s}} \nabla_{\phi m} \tilde{\eta}_{\bar{s}l} - \nabla_\phi^{\bar{q}} R_0^\beta l_{\bar{q}m}) U_{\bar{\gamma}\bar{\alpha}}^{\bar{\mu}} + U_{ml}^\beta (g_\phi^{\bar{\mu}s} \nabla_{\phi\bar{\gamma}} \tilde{\eta}_{s\bar{\alpha}} - \nabla_\phi^q R_0^{\bar{\mu}} \bar{\alpha}_{q\bar{\gamma}})) \\ - U_{ml}^\beta U_{\bar{\gamma}\bar{\alpha}}^{\bar{\mu}} (\tilde{\eta}_{p\bar{q}} g_\phi^{p\bar{\gamma}} g_\phi^{m\bar{q}} g_{\phi\bar{\mu}\beta} g_\phi^{l\bar{\alpha}} - g_\phi^{m\bar{\gamma}} \tilde{\eta}_{\bar{\mu}\beta} g_\phi^{l\bar{\alpha}} + g_\phi^{m\bar{\gamma}} g_{\phi\bar{\mu}\beta} g_\phi^{p\bar{\alpha}} g_\phi^{l\bar{q}} \tilde{\eta}_{p\bar{q}}) - \gamma S \leq C_1(S+1),$$

where the constant C_1 depends only on N , γ , ω_0 and $\|\tilde{\eta}\|_{C^1(B_r(p))}$. On the other hand, since

$$\nabla_{\phi l} X^\beta = \nabla_{0l} X^\beta + X^k U_{lk}^\beta,$$

$$\nabla_{\phi m} \nabla_{\phi l} X^\beta = \nabla_{0m} \nabla_{0l} X^\beta - \nabla_{0p} X^\beta \cdot U_{ml}^p + \nabla_{0l} X^p \cdot U_{pm}^\beta + \nabla_{\phi m} X^k \cdot U_{lk}^\beta + X^k \nabla_{\phi m} U_{lk}^\beta,$$

in the same way as [PSSW11, Section 6], we observe that

$$|(X; \text{III})| + |(X; \text{IV})| + |(X; \text{V})| \leq C_2 S |\nabla_\phi X|_{\omega_\phi},$$

$$\begin{aligned} |(X; \text{I})| + |(X; \text{II})| &\leq C_3(S+1) + S |\nabla_\phi X|_{\omega_\phi} + |X|_{\omega_\phi} |U|_{\omega_\phi} |\nabla_\phi U|_{\omega_\phi} \\ &\leq C_3(S+1) + S |\nabla_\phi X|_{\omega_\phi} + \frac{1}{2} |\nabla_\phi U|_{\omega_\phi}^2 + \frac{1}{2} |X|_{\omega_\phi}^2 |U|_{\omega_\phi}^2 \\ &\leq C_4(S+1) + \frac{1}{2} |\nabla_\phi U|_{\omega_\phi}^2 + S |\nabla_\phi X|_{\omega_\phi}, \end{aligned}$$

where C_4 depends only on X , ω_0 and N . Thus we have

$$(4.8) \quad \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) S \leq -\frac{1}{2} |\nabla_\phi U|_{\omega_\phi}^2 - |\bar{\nabla}_\phi U|_{\omega_\phi}^2 + (C_2 + 1) S |\nabla_\phi X|_{\omega_\phi} + (C_1 + C_4)(S+1).$$

On the other hand, the evolution equation of $|X|_{\omega_\phi}^2$ can be estimated as

$$(4.9) \quad \begin{aligned} \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) |X|_{\omega_\phi}^2 &= \gamma |X|_{\omega_\phi}^2 - |\nabla_\phi X|_{\omega_\phi}^2 + (\tilde{\eta}_{i\bar{j}} + \nabla_{\phi i} X_{\bar{j}}) X^i X^{\bar{j}} \\ &\leq -\frac{1}{2} |\nabla_\phi X|_{\omega_\phi}^2 + C_5. \end{aligned}$$

Now we work in local normal coordinates (z^i) with respect to ω_0 where ω_ϕ is diagonal. Since

$$0 \leq \text{tr} h \leq nN,$$

$$g_0^{j\bar{s}} g_\phi^{p\bar{q}} g_\phi^{m\bar{k}} \phi_{j\bar{k}p} \phi_{\bar{s}m\bar{q}} \geq \frac{1}{N} S,$$

$$|g_0^{i\bar{j}} \nabla_{\phi i} X_{\bar{j}}| \leq \text{tr} h \cdot |\text{tr} \nabla_\phi X| \leq C_6(S^{1/2} + 1) \leq \frac{1}{N+1} S + C_7,$$

we observe that

$$(4.10) \quad \begin{aligned} \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) \text{tr} h &= \gamma \text{tr} h + g_0^{i\bar{j}} (\tilde{\eta}_{i\bar{j}} + \nabla_{\phi i} X_{\bar{j}}) - g_\phi^{p\bar{q}} g_0^{\beta\bar{\gamma}} g_{\phi\alpha\bar{\gamma}} R_0^\alpha \beta_{\bar{q}p} - g_0^{j\bar{s}} g_\phi^{p\bar{q}} g_\phi^{m\bar{k}} \phi_{j\bar{k}p} \phi_{\bar{s}m\bar{q}} \\ &\leq C_8 - \frac{1}{N(N+1)} S. \end{aligned}$$

Let $r > r_1 > r/2$ and κ be a nonnegative smooth cut-off function that is identically equal to 1 on $\overline{B_{r_1}(p)}$ and vanishes on the outside of $B_r(p)$. Furthermore, we assume that

$$|\partial \kappa|_{\omega_0}, \quad |\sqrt{-1} \partial \bar{\partial} \kappa|_{\omega_0} \leq C_9.$$

We consider the function

$$W := \kappa^2 \frac{S}{K - |X|_{\omega_\phi}^2} + A \text{tr} h,$$

where K is a uniform constant such that $\frac{256}{257}K \leq K - |X|_{\omega_\phi}^2 \leq K$ and A is a uniform constant determined later. A direct computation shows that

$$\begin{aligned} \left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) W &= (-\Delta_{\omega_\phi} \kappa^2) \frac{S}{K - |X|_{\omega_\phi}^2} - 4\operatorname{Re} \left(\frac{\kappa \nabla_\phi \kappa}{K - |X|_{\omega_\phi}^2}, \nabla_\phi S \right)_{\omega_\phi} \\ &\quad - 4\operatorname{Re} \left(\kappa \nabla_\phi \kappa, \frac{S \cdot \nabla_\phi |X|_{\omega_\phi}^2}{(K - |X|_{\omega_\phi}^2)^2} \right)_{\omega_\phi} + \frac{\kappa^2}{K - |X|_{\omega_\phi}^2} \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) S \\ &\quad + \frac{\kappa^2 S}{(K - |X|_{\omega_\phi}^2)^2} \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) |X|_{\omega_\phi}^2 - \frac{2\kappa^2 S (\nabla_\phi |X|_{\omega_\phi}^2)^2}{(K - |X|_{\omega_\phi}^2)^3} \\ &\quad - \frac{2\kappa^2 \operatorname{Re}(\nabla_\phi |X|_{\omega_\phi}^2, \nabla_\phi S)_{\omega_\phi}}{(K - |X|_{\omega_\phi}^2)^2} + A \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) \operatorname{tr} h. \end{aligned}$$

Using (4.8), (4.9) and the facts

$$(4.11) \quad |\nabla_\phi |X|_{\omega_\phi}^2|_{\omega_\phi} \leq |X|_{\omega_\phi} |\nabla_\phi X|_{\omega_\phi},$$

$$(4.12) \quad |\nabla_\phi S|_{\omega_\phi}^2 \leq 2S(|\nabla_\phi U|_{\omega_\phi}^2 + |\overline{\nabla}_\phi U|_{\omega_\phi}^2),$$

we observe that

$$\begin{aligned} \left| (-\Delta_{\omega_\phi} \kappa^2) \frac{S}{K - |X|_{\omega_\phi}^2} \right| &\leq C_{10} S, \\ \left| 4\operatorname{Re} \left(\frac{\kappa \nabla_\phi \kappa}{K - |X|_{\omega_\phi}^2}, \nabla_\phi S \right)_{\omega_\phi} \right| &\leq \frac{4\sqrt{2}}{K - |X|_{\omega_\phi}^2} \kappa |\nabla_\phi \kappa|_{\omega_\phi} S^{1/2} (|\nabla_\phi U|_{\omega_\phi}^2 + |\overline{\nabla}_\phi U|_{\omega_\phi}^2)^{1/2} \\ &\leq C_{11} S + \frac{\kappa^2}{4(K - |X|_{\omega_\phi}^2)} (|\nabla_\phi U|_{\omega_\phi}^2 + |\overline{\nabla}_\phi U|_{\omega_\phi}^2), \\ \left| 4\operatorname{Re} \left(\kappa \nabla_\phi \kappa, \frac{S \cdot \nabla_\phi |X|_{\omega_\phi}^2}{(K - |X|_{\omega_\phi}^2)^2} \right)_{\omega_\phi} \right| &\leq C_{12} S + \frac{\kappa^2 S |\nabla_\phi X|_{\omega_\phi}^2}{4(K - |X|_{\omega_\phi}^2)^2}, \\ \frac{\kappa^2}{K - |X|_{\omega_\phi}^2} \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) S &\leq -\frac{\kappa^2}{2(K - |X|_{\omega_\phi}^2)} (|\nabla_\phi U|_{\omega_\phi}^2 + |\overline{\nabla}_\phi U|_{\omega_\phi}^2) + \frac{(C_2 + 1)\kappa^2 S |\nabla_\phi X|_{\omega_\phi}}{K - |X|_{\omega_\phi}^2} \\ &\quad + \frac{\kappa^2 (C_1 + C_4)}{K - |X|_{\omega_\phi}^2} (S + 1) \\ &\leq -\frac{\kappa^2}{2(K - |X|_{\omega_\phi}^2)} (|\nabla_\phi U|_{\omega_\phi}^2 + |\overline{\nabla}_\phi U|_{\omega_\phi}^2) + \frac{\kappa^2 S |\nabla_\phi X|_{\omega_\phi}^2}{8(K - |X|_{\omega_\phi}^2)^2} \\ &\quad + C_{13} (S + 1), \\ \frac{\kappa^2 S}{(K - |X|_{\omega_\phi}^2)^2} \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) |X|_{\omega_\phi}^2 &\leq -\frac{\kappa^2 S |\nabla_\phi X|_{\omega_\phi}^2}{2(K - |X|_{\omega_\phi}^2)^2} + C_{14} S, \end{aligned}$$

$$\begin{aligned}
\left| \frac{2\kappa^2 \operatorname{Re}(\nabla_\phi |X|_{\omega_\phi}^2, \nabla_\phi S)_{\omega_\phi}}{(K - |X|_{\omega_\phi}^2)^2} \right| &\leq \frac{2\sqrt{2}\kappa^2}{(K - |X|_{\omega_\phi}^2)^2} |X|_{\omega_\phi} |\nabla_\phi X|_{\omega_\phi} S^{1/2} (|\nabla_\phi U|_{\omega_\phi}^2 + |\bar{\nabla}_\phi U|_{\omega_\phi}^2)^{1/2} \\
&\leq \frac{\kappa^2 S |\nabla_\phi X|_{\omega_\phi}^2}{16(K - |X|_{\omega_\phi}^2)^2} + \frac{32\kappa^2 |X|_{\omega_\phi}^2}{(K - |X|_{\omega_\phi}^2)^2} (|\nabla_\phi U|_{\omega_\phi}^2 + |\bar{\nabla}_\phi U|_{\omega_\phi}^2) \\
&\leq \frac{\kappa^2 S |\nabla_\phi X|_{\omega_\phi}^2}{16(K - |X|_{\omega_\phi}^2)^2} + \frac{\kappa^2}{8(K - |X|_{\omega_\phi}^2)} (|\nabla_\phi U|_{\omega_\phi}^2 + |\bar{\nabla}_\phi U|_{\omega_\phi}^2) \\
&\quad (\text{because } \frac{256}{257}K < K - |X|_{\omega_\phi}^2 < K).
\end{aligned}$$

Hence, combining with (4.10), we get

$$\left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) W \leq \left(C_{10} + C_{11} + C_{13} + C_{14} - \frac{A}{N(N+1)} \right) S + C_{13}.$$

Let (x_0, t_0) be the maximum point of W on $\overline{B_r(p)} \times [0, T]$. If $t_0 = 0$, then S is bounded by the initial data $\|\phi(\cdot, 0)\|_{C^3(B_r(p))}$. Moreover, we find that $W \equiv \operatorname{Atr} h$ on the boundary of $B_r(p)$ where the function $\operatorname{tr} h$ is uniformly controlled. Then we may assume that $t_0 > 0$ and x_0 does not lie in the boundary of $B_r(p)$. By the maximum principle, we have

$$0 \leq \left(C_{10} + C_{11} + C_{13} + C_{14} - \frac{A}{N(N+1)} \right) S(x_0, t_0) + C_{13}.$$

Taking $A := N(N+1)(C_{10} + C_{11} + C_{13} + C_{14} + 1)$, we conclude that $S(x_0, t_0) \leq C_{13}$. Since $0 \leq \operatorname{tr} h \leq nN$, we have

$$S \leq \frac{257}{256} C_{13} + AnNK \leq C_{15}$$

on $\overline{B_{r_1}(p)} \times [0, T]$, where the constant C_{15} depends only on $N, \gamma, \omega_0, X, \|\phi(\cdot, 0)\|_{C^3(B_r(p))}$ and $\|\tilde{\eta}\|_{C^1(B_r(p))}$. In particular, $|\nabla_\phi X|_{\omega_\phi}^2$ is uniformly bounded.

Next, we establish the uniform bound of $|\operatorname{Rm}_\phi|_{\omega_\phi}^2$. The evolution equation of the full curvature tensor along MTKRF is

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) R_{\phi\bar{j}i\bar{l}k} &= R_{\phi\bar{j}i}{}^{p\bar{q}} R_{\phi\bar{l}k\bar{q}p} + R_{\phi\bar{l}i}{}^{p\bar{q}} R_{\phi\bar{j}k\bar{q}p} - R_{\phi\bar{j}p\bar{l}}{}^{\bar{q}} R_{\phi}{}^p{}_{i\bar{q}k} - R_{\phi p\bar{l}} R_{\phi\bar{j}i}{}^p{}_k \\
&\quad - R_{\phi\bar{j}h} R_{\phi}{}^h{}_{i\bar{l}k} - \nabla_{\phi\bar{l}} \nabla_{\phi k} \tilde{\eta}_{i\bar{j}} + \gamma R_{\phi\bar{j}i\bar{l}k} - \tilde{\eta}_{j\bar{h}} R_{\phi}{}^h{}_{i\bar{k}\bar{l}} \\
&\quad - \underbrace{\nabla_{\phi\bar{l}} \nabla_{\phi k} \nabla_{\phi i} X_{\bar{j}} - \nabla_{\phi h} X_{\bar{j}} \cdot R_{\phi}{}^h{}_{i\bar{k}\bar{l}}}_{\text{additional terms arising from } X}.
\end{aligned} \tag{4.13}$$

By direct computations, we get

$$(4.14) \quad \nabla_{\phi\bar{l}} \nabla_{\phi k} \tilde{\eta}_{i\bar{j}} = \nabla_{0\bar{l}} \nabla_{0k} \tilde{\eta}_{i\bar{j}} - U_{\bar{l}j}^{\bar{s}} \nabla_{0k} \tilde{\eta}_{i\bar{s}} - \nabla_{0\bar{l}} U_{ki}^s \tilde{\eta}_{s\bar{j}} - U_{ki}^s \nabla_{0\bar{l}} \tilde{\eta}_{s\bar{j}} + U_{ki}^s U_{\bar{l}j}^{\bar{t}} \tilde{\eta}_{s\bar{t}},$$

$$(4.15) \quad \nabla_{0\bar{k}} U_{jl}^i = \nabla_{\phi\bar{k}} U_{jl}^i = \partial_{\bar{k}} U_{jl}^i = -R_{\phi}{}^i{}_{l\bar{k}j} + R_0{}^i{}_{l\bar{k}j},$$

$$(4.16) \quad \nabla_{\phi\bar{u}} \nabla_{\phi l} \nabla_{\phi j} X^i = -\nabla_{\phi l} X^k \cdot R_{\phi j}{}^i{}_{k\bar{u}} - X^k \nabla_{\phi l} R_{\phi j}{}^i{}_{k\bar{u}} - \nabla_{\phi j} X^p \cdot R_{\phi p}{}^i{}_{l\bar{u}} + \nabla_{\phi s} X^i \cdot R_{\phi l}{}^s{}_{j\bar{u}}.$$

Hence, using the uniform bound of S , $|X|_{\omega_\phi}^2$ and $|\nabla_\phi X|_{\omega_\phi}^2$, we have

$$\left| \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) \text{Rm}_\phi \right|_{\omega_\phi} \leq C_{16}(|\text{Rm}_\phi|_{\omega_\phi}^2 + |\text{Rm}_\phi|_{\omega_\phi} + 1) + C_{17}|\nabla_\phi \text{Rm}_\phi|_{\omega_\phi}.$$

Thus, by the uniform bound of $|\nabla_\phi X|_{\omega_\phi}^2$ and the equation (4.2), we obtain

$$\begin{aligned} \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) |\text{Rm}_\phi|_{\omega_\phi}^2 &\leq C_{18}(|\text{Rm}_\phi|_{\omega_\phi}^3 + |\text{Rm}_\phi|_{\omega_\phi}^2) + 2 \left| \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) \text{Rm}_\phi \right|_{\omega_\phi} |\text{Rm}_\phi|_{\omega_\phi} \\ &\quad - |\nabla_\phi \text{Rm}_\phi|_{\omega_\phi}^2 - |\bar{\nabla}_\phi \text{Rm}_\phi|_{\omega_\phi}^2 \\ (4.17) \quad &\leq C_{19}(|\text{Rm}_\phi|_{\omega_\phi}^3 + 1) - \frac{1}{2} |\nabla_\phi \text{Rm}_\phi|_{\omega_\phi}^2 - |\bar{\nabla}_\phi \text{Rm}_\phi|_{\omega_\phi}^2. \end{aligned}$$

Now we take a smaller radius r_2 satisfying $r_1 > r_2 > r/2$ and show that $|\text{Rm}_\phi|_{\omega_\phi}^2$ is uniformly bounded on $\overline{B_{r_2}(p)}$. Let μ be a nonnegative smooth cut-off function that is identically equal to 1 on $\overline{B_{r_2}(p)}$, vanishes on the outside of $B_{r_1}(p)$ and satisfies

$$|\partial\mu|_{\omega_0}, \quad |\sqrt{-1}\partial\bar{\partial}\mu|_{\omega_0} \leq C_{20}.$$

Let L be a uniform constant satisfying $\frac{512}{513}L \leq L - S \leq L$. We consider the function

$$G := \mu^2 \frac{|\text{Rm}_\phi|_{\omega_\phi}^2}{L - S} + BS,$$

where B is a uniform constant determined later. By computing, we have

$$\begin{aligned} \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) G &= (-\Delta_{\omega_\phi} \mu^2) \frac{|\text{Rm}_\phi|_{\omega_\phi}^2}{L - S} - 4\text{Re} \left(\frac{\mu \nabla_\phi \mu}{L - S}, \nabla_\phi |\text{Rm}_\phi|_{\omega_\phi}^2 \right)_{\omega_\phi} \\ &\quad - 4\text{Re} \left(\mu \nabla_\phi \mu, \frac{|\text{Rm}_\phi|_{\omega_\phi}^2 \nabla_\phi S}{(L - S)^2} \right)_{\omega_\phi} + \frac{\mu^2}{L - S} \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) |\text{Rm}_\phi|_{\omega_\phi}^2 \\ &\quad + \frac{\mu^2 |\text{Rm}_\phi|_{\omega_\phi}^2}{(L - S)^2} \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) S - \frac{2\mu^2 |\text{Rm}_\phi|_{\omega_\phi}^2}{(L - S)^3} |\nabla_\phi S|_{\omega_\phi}^2 \\ &\quad - 2\text{Re} \left(\mu^2 \frac{\nabla_\phi S}{(L - S)^2}, \nabla_\phi |\text{Rm}_\phi|_{\omega_\phi}^2 \right)_{\omega_\phi} + B \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) S. \end{aligned}$$

Then, by (4.8), (4.12), (4.17) and

$$(4.18) \quad |\nabla_\phi |\text{Rm}_\phi|_{\omega_\phi}^2|_{\omega_\phi} \leq |\text{Rm}_\phi|_{\omega_\phi} (|\nabla_\phi \text{Rm}_\phi|_{\omega_\phi} + |\bar{\nabla}_\phi \text{Rm}_\phi|_{\omega_\phi}),$$

we know that

$$\begin{aligned} \left| (-\Delta_{\omega_\phi} \mu^2) \frac{|\text{Rm}_\phi|_{\omega_\phi}^2}{L - S} \right| &\leq C_{21} |\text{Rm}_\phi|_{\omega_\phi}^2, \\ \left| 4\text{Re} \left(\frac{\mu \nabla_\phi \mu}{L - S}, \nabla_\phi |\text{Rm}_\phi|_{\omega_\phi}^2 \right)_{\omega_\phi} \right| &\leq \frac{4}{L - S} \mu |\nabla_\phi \mu|_{\omega_\phi} |\text{Rm}_\phi|_{\omega_\phi} (|\nabla_\phi \text{Rm}_\phi|_{\omega_\phi} + |\bar{\nabla}_\phi \text{Rm}_\phi|_{\omega_\phi}) \\ &\leq C_{22} |\text{Rm}_\phi|_{\omega_\phi}^2 + \frac{\mu^2}{4(L - S)} (|\nabla_\phi \text{Rm}_\phi|_{\omega_\phi}^2 + |\bar{\nabla}_\phi \text{Rm}_\phi|_{\omega_\phi}^2), \end{aligned}$$

$$\begin{aligned}
\left| 4\operatorname{Re} \left(\mu \nabla_\phi \mu, \frac{|\operatorname{Rm}_\phi|_{\omega_\phi}^2 \nabla_\phi S}{(L-S)^2} \right)_{\omega_\phi} \right| &\leq \frac{4\sqrt{2}|\operatorname{Rm}_\phi|_{\omega_\phi}^2}{(L-S)^2} \mu |\nabla_\phi \mu|_{\omega_\phi} S^{1/2} (|\nabla_\phi U|_{\omega_\phi}^2 + |\bar{\nabla}_\phi U|_{\omega_\phi}^2)^{1/2} \\
&\leq C_{23} |\operatorname{Rm}_\phi|_{\omega_\phi}^2 + \frac{\mu^2 |\operatorname{Rm}_\phi|_{\omega_\phi}^2}{4(L-S)^2} (|\nabla_\phi U|_{\omega_\phi}^2 + |\bar{\nabla}_\phi U|_{\omega_\phi}^2), \\
\frac{\mu^2}{L-S} \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) |\operatorname{Rm}_\phi|_{\omega_\phi}^2 &\leq \frac{C_{19}\mu^2}{L-S} |\operatorname{Rm}_\phi|_{\omega_\phi}^3 - \frac{\mu^2}{2(L-S)} (|\nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi}^2 + |\bar{\nabla}_\phi \operatorname{Rm}_\phi|_{\omega_\phi}^2) + C_{24} \\
&\leq \frac{\mu^2 |\operatorname{Rm}_\phi|_{\omega_\phi}^4}{8(L-S)^2} + C_{25}\mu^2 |\operatorname{Rm}_\phi|_{\omega_\phi}^2 - \frac{\mu^2}{2(L-S)} (|\nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi}^2 + |\bar{\nabla}_\phi \operatorname{Rm}_\phi|_{\omega_\phi}^2) \\
&\quad + C_{24} \\
&\leq C_{26} |\operatorname{Rm}_\phi|_{\omega_\phi}^2 + \frac{\mu^2 |\operatorname{Rm}_\phi|_{\omega_\phi}^2}{8(L-S)^2} (|\nabla_\phi U|_{\omega_\phi}^2 + |\bar{\nabla}_\phi U|_{\omega_\phi}^2) \\
&\quad - \frac{\mu^2}{2(L-S)} (|\nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi}^2 + |\bar{\nabla}_\phi \operatorname{Rm}_\phi|_{\omega_\phi}^2) + C_{24} \\
&\quad \text{(where we used (4.15) in the last inequality),} \\
\frac{\mu^2 |\operatorname{Rm}_\phi|_{\omega_\phi}^2}{(L-S)^2} \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) S &\leq C_{27} |\operatorname{Rm}_\phi|_{\omega_\phi}^2 - \frac{\mu^2 |\operatorname{Rm}_\phi|_{\omega_\phi}^2}{2(L-S)^2} (|\nabla_\phi U|_{\omega_\phi}^2 + |\bar{\nabla}_\phi U|_{\omega_\phi}^2), \\
\left| 2\operatorname{Re} \left(\mu^2 \frac{\nabla_\phi S}{(L-S)^2}, \nabla_\phi |\operatorname{Rm}_\phi|_{\omega_\phi}^2 \right)_{\omega_\phi} \right| &\leq \frac{2\sqrt{2}\mu^2}{(L-S)^2} S^{1/2} (|\nabla_\phi U|_{\omega_\phi}^2 + |\bar{\nabla}_\phi U|_{\omega_\phi}^2)^{1/2} \cdot \\
&\quad |\operatorname{Rm}_\phi|_{\omega_\phi} (|\nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi} + |\bar{\nabla}_\phi \operatorname{Rm}_\phi|_{\omega_\phi}) \\
&\leq \frac{64\mu^2 S}{(L-S)^2} (|\nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi}^2 + |\bar{\nabla}_\phi \operatorname{Rm}_\phi|_{\omega_\phi}^2) \\
&\quad + \frac{\mu^2 |\operatorname{Rm}_\phi|_{\omega_\phi}^2}{16(L-S)^2} (|\nabla_\phi U|_{\omega_\phi}^2 + |\bar{\nabla}_\phi U|_{\omega_\phi}^2) \\
&\leq \frac{\mu^2}{8(L-S)} (|\nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi}^2 + |\bar{\nabla}_\phi \operatorname{Rm}_\phi|_{\omega_\phi}^2) \\
&\quad + \frac{\mu^2 |\operatorname{Rm}_\phi|_{\omega_\phi}^2}{16(L-S)^2} (|\nabla_\phi U|_{\omega_\phi}^2 + |\bar{\nabla}_\phi U|_{\omega_\phi}^2) \\
&\quad \text{(because } \frac{512}{513}L < L-S < L\text{).}
\end{aligned}$$

As in the previous part, we may only consider an inner point (x_0, t_0) which is a maximum point of G achieved on $\overline{B_{r_1}(p)} \times [0, T]$. By the maximum principle, we have

$$0 \leq \left(C_{21} + C_{22} + C_{23} + C_{26} + C_{27} - \frac{B}{2} \right) |\operatorname{Rm}_\phi|_{\omega_\phi}^2(x_0, t_0) + C_{28}.$$

Now we set $B := 2(C_{21} + C_{22} + C_{23} + C_{26} + C_{27} + 1)$. Then we obtain

$$|\operatorname{Rm}_\phi|_{\omega_\phi}^2(x_0, t_0) \leq C_{28}.$$

Since S is uniformly bounded, this implies

$$|\operatorname{Rm}_\phi|_{\omega_\phi}^2 \leq C_{29}$$

on $\overline{B_{r_2}(p)} \times [0, T]$, where C_{29} depends only on $N, \gamma, \omega_0, X, \|\phi(\cdot, 0)\|_{C^4(B_r(p))}$ and $\|\tilde{\eta}\|_{C^2(B_r(p))}$.

Following [LZ17], we say that ϕ is $C^{k,\alpha}$ if its $C^{k,\alpha}$ norm can be controlled by a constant depending only on $N, \gamma, \omega_0, X, \|\phi(\cdot, 0)\|_{C^{k+1}(B_r(p))}, \|\phi\|_{C^0(B_r(p) \times [0, T])}, \|\tilde{\eta}\|_{C^{k-1}(B_r(p))}$ and $\|F\|_{C^0(B_r(p))}$. Likewise, we say that $\dot{\phi}$ is $C^{k,\alpha}$ if its $C^{k,\alpha}$ norm can be controlled by a constant depending only on $N, \gamma, \omega_0, X, \|\phi(\cdot, 0)\|_{C^{k+3}(B_r(p))}, \|\phi\|_{C^0(B_r(p) \times [0, T])}, \|\tilde{\eta}\|_{C^{k+1}(B_r(p))}$ and $\|F\|_{C^0(B_r(p))}$. Since $|\text{Rm}_\phi|_{\omega_\phi}^2$ and $|\nabla_\phi X|_{\omega_\phi}^2$ are uniformly bounded, we know that $\dot{\phi}$ is $C^{1,\alpha}$. Differentiating the equation (4.3) with respect to z^k , we get

$$\frac{d}{dt} \frac{\partial \phi}{\partial z^k} = (\Delta_{\omega_\phi} + X) \frac{\partial \phi}{\partial z^k} + g_\phi^{i\bar{j}} \frac{\partial g_{0i\bar{j}}}{\partial z^k} - g_0^{i\bar{j}} \frac{\partial g_{0i\bar{j}}}{\partial z^k} + \frac{\partial F}{\partial z^k} + \gamma \frac{\partial \phi}{\partial z^k} + \frac{\partial \theta_X}{\partial z^k} + \frac{\partial X^i}{\partial z^k} \frac{\partial \phi}{\partial z^i}.$$

From the above Calabi's C^3 -estimate, we know that ϕ is $C^{2,\alpha}$ and then the coefficients of Δ_{ω_ϕ} are $C^{0,\alpha}$. Since F is the twisted Ricci potential, taking the trace with respect to ω_0 yields

$$\Delta_{\omega_0} F = -\text{tr}_{\omega_0} \text{Ric}(\omega_0) + \gamma + \text{tr}_{\omega_0} \tilde{\eta}.$$

Hence the $C^{1,\alpha}$ -norm of F on $B_{r_2}(p)$ only depends on $\omega_0, \|\tilde{\eta}\|_{C^0(B_r(p))}$ and $\|F\|_{C^0(B_r(p))}$. By the standard elliptic Schauder estimates, we conclude that ϕ is $C^{3,\alpha}$ on $B_{r_3}(p) \times [0, T]$, where $r_2 > r_3 > r/2$.

Now we prove that $|\nabla_\phi \text{Rm}_\phi|_{\omega_\phi}^2$ is uniformly bounded. First we compute the evolution equation of U as

$$(4.19) \quad \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) U_{ml}^\beta = \nabla_{\phi m} (\tilde{\eta}^\beta_l + \nabla_{\phi l} X^\beta) - \nabla_\phi^{\bar{q}} R_0^\beta{}_{l\bar{q}m}.$$

Since $\tilde{\eta}, \text{Rm}_0$ and X are t -independent tensors, we know that

$$(4.20) \quad |\nabla_\phi \tilde{\eta}|_{\omega_\phi} \leq C_{30},$$

$$(4.21) \quad |\nabla_\phi^2 \tilde{\eta}|_{\omega_\phi} + |\nabla_\phi^2 \text{Rm}_0|_{\omega_\phi} + |\nabla_\phi^2 X|_{\omega_\phi} \leq C_{31}(1 + |\nabla_\phi U|_{\omega_\phi}),$$

$$|\nabla_\phi^3 X|_{\omega_\phi} \leq C_{32}(1 + |\nabla_\phi U|_{\omega_\phi} + |\nabla_\phi^2 U|_{\omega_\phi}).$$

On the other hand, by the Ricci identity, we have

$$\left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) \nabla_\phi U = \nabla_\phi \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) U + U * \nabla_\phi (\text{Rm}_\phi + \tilde{\eta} + \nabla_\phi X) + \text{Rm}_\phi * \nabla_\phi U,$$

where $*$ means the general pairs of tensors. Thus we obtain

$$(4.22) \quad \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) |\nabla_\phi U|_{\omega_\phi}^2 \leq C_{33}(|\nabla_\phi U|_{\omega_\phi}^2 + 1) + |\nabla_\phi \text{Rm}_\phi|_{\omega_\phi}^2 - \frac{1}{2} |\nabla_\phi \nabla_\phi U|_{\omega_\phi}^2 - |\bar{\nabla}_\phi \nabla_\phi U|_{\omega_\phi}^2.$$

Now we set $r_3 > r'_3 > r/2$ and take a smooth cut-off function ϱ such that

$$|\partial \varrho|_{\omega_0}, |\sqrt{-1} \partial \bar{\partial} \varrho|_{\omega_0} \leq C_{34},$$

and set

$$I := \varrho^2 |\nabla_\phi U|_{\omega_\phi}^2 + ES + 2|\text{Rm}_\phi|_{\omega_\phi}^2,$$

where E is a uniform constant determined later. Then we see that

$$\begin{aligned} \left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) I &\leq (-\Delta_{\omega_\phi} \varrho^2) |\nabla_\phi U|_{\omega_\phi}^2 - 4\operatorname{Re}(\varrho \nabla_\phi \varrho, \nabla_\phi |\nabla_\phi U|_{\omega_\phi}^2)_{\omega_\phi} \\ &\quad + \varrho^2 \left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) |\nabla_\phi U|_{\omega_\phi}^2 + E \left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) S + 2 \left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) |\operatorname{Rm}_\phi|_{\omega_\phi}^2. \end{aligned}$$

The first and second term of the RHS are estimated as

$$|(-\Delta_{\omega_\phi} \varrho^2) |\nabla_\phi U|_{\omega_\phi}^2| \leq C_{35} |\nabla_\phi U|_{\omega_\phi}^2,$$

$$|4\operatorname{Re}(\varrho \nabla_\phi \varrho, \nabla_\phi |\nabla_\phi U|_{\omega_\phi}^2)_{\omega_\phi}| \leq C_{36} |\nabla_\phi U|_{\omega_\phi}^2 + \frac{\varrho^2}{4} (|\nabla_\phi \nabla_\phi U|_{\omega_\phi}^2 + |\bar{\nabla}_\phi \nabla_\phi U|_{\omega_\phi}^2).$$

Thus, combining with (4.8) and (4.17), we obtain

$$\left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) I \leq \left(C_{33} + C_{35} + C_{36} - \frac{E}{2}\right) |\nabla_\phi U|_{\omega_\phi}^2 + C_{37}.$$

Hence, if we set $E := 2(C_{33} + C_{35} + C_{36} + 1)$, the maximum principle implies the uniform bound of $|\nabla_\phi U|_{\omega_\phi}^2$ on $\overline{B_{r'_3}(p)} \times [0, T]$. Let D denote the real covariant derivative with respect to ω_ϕ (extended linearly on the space of complex tensors). Combining with the uniform bound of $|\operatorname{Rm}_\phi|_{\omega_\phi}^2$ and (4.15), we have

$$|DU|_{\omega_\phi}^2 \leq C_{38}$$

on $\overline{B_{r'_3}(p)} \times [0, T]$, where the constant C_{38} depends only on $N, \gamma, \omega_0, X, \|\phi(\cdot, 0)\|_{C^4(B_r(P))}$ and $\|\tilde{\eta}\|_{C^2(B_r(p))}$. In particular, we find that $|D^2 X|_{\omega_\phi}^2$ is uniformly bounded. Applying ∇_ϕ to (4.13), we see that

$$\left| \nabla_\phi \left(\frac{d}{dt} - \Delta_{\omega_\phi} \right) \operatorname{Rm}_\phi \right|_{\omega_\phi} \leq C_{39} (|\nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi} + |\nabla_\phi \bar{\nabla}_\phi \nabla_\phi \tilde{\eta}|_{\omega_\phi} + |\nabla_\phi \bar{\nabla}_\phi \nabla_\phi^2 X|_{\omega_\phi}).$$

Applying ∇_ϕ to (4.14) and (4.16), and using the uniform bound of $|DU|_{\omega_\phi}^2$, we have

$$\begin{aligned} |\nabla_\phi \bar{\nabla}_\phi \nabla_\phi \tilde{\eta}|_{\omega_\phi} &\leq C_{40} (1 + |\nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi}), \\ |\nabla_\phi \bar{\nabla}_\phi \nabla_\phi^2 X|_{\omega_\phi} &\leq C_{41} (1 + |\nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi} + |\nabla_\phi^2 \operatorname{Rm}_\phi|_{\omega_\phi}). \end{aligned}$$

Combining with

$$\left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) \nabla_\phi \operatorname{Rm}_\phi = \nabla_\phi \left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) \operatorname{Rm}_\phi + \operatorname{Rm}_\phi * \nabla_\phi (\operatorname{Rm}_\phi + \tilde{\eta} + \nabla_\phi X),$$

we find that

$$\left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) |\nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi}^2 \leq C_{42} (|\nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi}^2 + 1) - \frac{1}{2} |\nabla_\phi \nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi}^2 - |\bar{\nabla}_\phi \nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi}^2.$$

Now we take a smaller radius $r'_3 > r''_3 > r/2$ and a smooth cut-off function σ that is identically equal to 1 on $\overline{B_{r''_3}(p)}$, vanishes on the outside of $B_{r'_3}(p)$ and satisfies

$$|\partial \sigma|_{\omega_0}, |\sqrt{-1} \partial \bar{\partial} \sigma|_{\omega_0} \leq C_{43}.$$

We apply the maximum principle to the function $\sigma^2 |\nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi}^2 + P |\operatorname{Rm}_\phi|_{\omega_\phi}^2$ (where P is a suitable uniform constant). Then, as in the previous argument, we find that $|\nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi}^2$ is uniformly bounded on $\overline{B_{r''_3}(p)} \times [0, T]$. Thus we have

$$|D \operatorname{Rm}_\phi|_{\omega_\phi}^2 \leq C_{44}$$

on $\overline{B_{r_3''}(p)} \times [0, T]$, where C_{44} depends only on N , γ , ω_0 , X , $\|\phi(\cdot, 0)\|_{C^5(B_r(p))}$ and $\|\tilde{\eta}\|_{C^3(B_r(p))}$.

Applying D to the equation (4.2), we have

$$D\sqrt{-1}\partial\bar{\partial}\dot{\phi} = D\text{Ric}(\omega_\phi) + D\tilde{\eta} + D(\nabla_\phi X^\flat),$$

where $X_j^\flat := g_{\phi i \bar{j}} X^i$. Taking the trace, we have

$$\begin{aligned} |\Delta_{\omega_\phi} D\dot{\phi}|_{\omega_\phi} &\leq |D\Delta_{\omega_\phi} \dot{\phi}|_{\omega_\phi} + |D\text{Rm}_\phi * \dot{\phi}|_{\omega_\phi} + |\text{Rm}_\phi * D\dot{\phi}|_{\omega_\phi} \\ &\leq C_{45}(|D\text{Rm}_\phi|_{\omega_\phi} + |D\tilde{\eta}|_{\omega_\phi} + |D^2 X|_{\omega_\phi} + |D\text{Rm}_\phi|_{\omega_\phi} |\dot{\phi}| + |\text{Rm}_\phi|_{\omega_\phi} |D\dot{\phi}|_{\omega_\phi}) \end{aligned}$$

From the above computations and the fact that $\dot{\phi}$ is $C^{1,\alpha}$, we find that $D\dot{\phi}$ is $C^{1,\alpha}$, which implies that $\dot{\phi}$ is $C^{2,\alpha}$. Differentiating the equation (4.3) two times and using the elliptic Schauder estimates, we have ϕ is $C^{4,\alpha}$ on $B_{r_4}(p) \times [0, T]$, where $r_3'' > r_4 > r/2$.

Now we establish the $C^{k,\alpha}$ -estimate for ϕ . For this, we set the following induction hypothesis:

$$(H_k) \quad \begin{cases} |D^j \text{Rm}|_{\omega_\phi}^2 \leq C_j^1 \\ \dot{\phi} \text{ is } C^{j+1,\alpha} \\ \phi \text{ is } C^{j+3,\alpha} \end{cases} \quad \text{on } \overline{B_{r_{j+3}}(p)} \times [0, T] \text{ for all } j = 0, 1, \dots, k,$$

where $r > r_1 > \dots > r_{k+2} > r_{k+3} > r/2$ and the constant C_j^1 depends only on N , γ , ω_0 , X , $\|\phi(\cdot, 0)\|_{C^{j+4}(B_r(p))}$, $\|\phi\|_{C^0(B_r(p) \times [0, T])}$, $\|\tilde{\eta}\|_{C^{j+2}(B_r(p))}$ and $\|F\|_{C^0(B_r(p))}$. We have already seen that this statement is established for $k = 0, 1$. Now we assume that the induction hypothesis (H_k) holds for some $k \geq 1$. Since ϕ is $C^{k+3,\alpha}$, we observe that

$$|D^j U|_{\omega_\phi}^2 \leq C_{46} \quad \text{for } j = 0, 1, \dots, k.$$

In particular, for any t -independent tensor A , we find that $|D^j A|_{\omega_\phi}^2$ is uniformly bounded for $j = 0, 1, \dots, k+1$. We first show the uniform bound of $|D^{k+1} U|_{\omega_\phi}^2$. Let r, s ($r+s = k+1$) are non-negative integers. Then any $(k+1)$ -derivative of U differs from $\nabla_\phi^r \bar{\nabla}_\phi^s U$ by a linear combination of $D^i U * D^{r+s-2-i} \text{Rm}_\phi$ ($0 \leq i \leq r+s-2$), which has been already estimated by the induction hypothesis (H_k) . Thus we may only consider $\nabla_\phi^r \bar{\nabla}_\phi^s U$. Moreover, the equation (4.15) and (H_k) indicate that we should only consider $\nabla_\phi^{k+1} U$. Using the Ricci identity repeatedly, we have

$$\begin{aligned} \left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) \nabla_\phi^{k+1} U &= \underbrace{\nabla_\phi^{k+1} \left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) U}_{(\nabla^{k+1} U; \text{I})} + \underbrace{\sum_{\substack{p \geq 0, q \geq 1 \\ p+q=k+1}} \nabla_\phi^p U * \nabla_\phi^q (\text{Rm}_\phi + \tilde{\eta} + \nabla_\phi X)}_{(\nabla^{k+1} U; \text{II})} \\ &\quad + \underbrace{\sum_{\substack{p \geq 0, q \geq 1 \\ p+q=k+1}} \nabla_\phi^p \text{Rm}_\phi * \nabla_\phi^q U}_{(\nabla^{k+1} U; \text{III})}. \end{aligned}$$

By (4.19) and (H_k) , we observe that

$$|(\nabla^{k+1} U; \text{I})|_{\omega_\phi} \leq C_{47}(1 + |\nabla_\phi^{k+1} U|_{\omega_\phi} + |\nabla_\phi^{k+2} U|_{\omega_\phi}),$$

$$|(\nabla^{k+1}U; \text{II})|_{\omega_\phi} \leq C_{48}(1 + |\nabla_\phi^{k+1}\text{Rm}_\phi|_{\omega_\phi} + |\nabla_\phi^{k+1}U|_{\omega_\phi}),$$

$$|(\nabla^{k+1}U; \text{III})|_{\omega_\phi} \leq C_{49}(1 + |\nabla_\phi^{k+1}\text{Rm}_\phi|_{\omega_\phi}).$$

Thus the evolution equation of $|\nabla_\phi^{k+1}U|_{\omega_\phi}^2$ can be estimated as

$$(4.23) \quad \left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) |\nabla_\phi^{k+1}U|_{\omega_\phi}^2 \leq -\frac{1}{2}|\nabla_\phi^{k+2}U|_{\omega_\phi}^2 - |\bar{\nabla}_\phi \nabla_\phi^{k+1}U|_{\omega_\phi}^2 + C_{50}|\nabla_\phi^{k+1}U|_{\omega_\phi}^2 + |\nabla_\phi^{k+1}\text{Rm}_\phi|_{\omega_\phi}^2.$$

Hence we should compute the evolution equation of $|\nabla_\phi^k U|_{\omega_\phi}^2$ and $|\nabla_\phi^k \text{Rm}_\phi|_{\omega_\phi}^2$, and add them to the above equation. It is not hard to see that

$$(4.24) \quad \left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) |\nabla_\phi^k U|_{\omega_\phi}^2 \leq C_{51} - \frac{1}{2}|\nabla_\phi^{k+1}U|_{\omega_\phi}^2 - |\bar{\nabla}_\phi \nabla_\phi^k U|_{\omega_\phi}^2,$$

$$(4.25) \quad \left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) |\nabla_\phi^k \text{Rm}_\phi|_{\omega_\phi}^2 \leq C_{52} - \frac{1}{2}|\nabla_\phi^{k+1}\text{Rm}_\phi|_{\omega_\phi}^2 - |\bar{\nabla}_\phi \nabla_\phi^k \text{Rm}_\phi|_{\omega_\phi}^2.$$

Actually, we can compute the first item in the same way as (4.23). For the second item, one should refer to the computation of (4.28). Hence we take a smooth cut-off function ς and apply the maximum principle to the function $\varsigma^2 |\nabla_\phi^{k+1}U|_{\omega_\phi}^2 + Q|\nabla_\phi^k U|_{\omega_\phi}^2 + 2|\nabla_\phi^k \text{Rm}_\phi|_{\omega_\phi}^2$ (for a suitable uniform constant Q) to get the uniform control of $|\nabla_\phi^{k+1}U|_{\omega_\phi}^2$ in $\overline{B_{r'_{j+3}}(p)} \times [0, T]$ with a smaller radius $r_{k+3} > r'_{k+3} > r/2$. Thus we have

$$|D^{k+1}U|_{\omega_\phi}^2 \leq C_{53}$$

on $\overline{B_{r'_{j+3}}(p)} \times [0, T]$, where the constant C_{53} depends only on $N, \gamma, \omega_0, X, \|\phi(\cdot, 0)\|_{C^{k+4}(B_r(p))}, \|\phi\|_{C^0(B_r(p) \times [0, T])}, \|\tilde{\eta}\|_{C^{k+2}(B_r(p))}$ and $\|F\|_{C^0(B_r(p))}$. In particular, we find that $|D^{k+2}X|_{\omega_\phi}^2$ is uniformly bounded.

Next, we establish the uniform estimate for $|D^{k+1}\text{Rm}_\phi|_{\omega_\phi}^2$. As in the previous case, we may only consider the tensor of the form $\nabla_\phi^r \bar{\nabla}_\phi^s \text{Rm}_\phi$ for non-negative integers r, s such that $r + s = k + 1$. Moreover, by the symmetries of Rm_ϕ , we may also assume that $r \neq 0$.

Case 1: $r, s \neq 0$.

Using the Ricci identity repeatedly, we have

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) \nabla_\phi^r \bar{\nabla}_\phi^s \text{Rm}_\phi &= \underbrace{\nabla_\phi^r \bar{\nabla}_\phi^s \text{Rm}_\phi \left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) \text{Rm}_\phi}_{(\nabla^r \bar{\nabla}^s \text{Rm}; \text{I})} \\
&+ \underbrace{\sum_{\substack{p \geq 0, q \geq 1 \\ p+q=k+1}} \nabla_\phi^p \bar{\nabla}_\phi^s \text{Rm}_\phi * \nabla_\phi^q (\text{Rm}_\phi + \tilde{\eta} + \nabla_\phi X)}_{(\nabla^r \bar{\nabla}^s \text{Rm}; \text{II})} \\
&+ \underbrace{\sum_{\substack{p \geq 0, q \geq 1 \\ p+q=k+1}} \nabla_\phi^p \text{Rm}_\phi * \nabla_\phi^q \bar{\nabla}_\phi^s (\text{Rm}_\phi + \tilde{\eta} + \nabla_\phi X)}_{(\nabla^r \bar{\nabla}^s \text{Rm}; \text{III})} \\
&+ \underbrace{\sum_{\substack{p \geq 0, q \geq 1 \\ p+q=k+1 \\ i=0,1,\dots,r}} \nabla_\phi^i \bar{\nabla}_\phi^p \text{Rm}_\phi * \nabla_\phi^{r-i} \bar{\nabla}_\phi^q (\text{Rm}_\phi + \tilde{\eta} + \nabla_\phi X)}_{(\nabla^r \bar{\nabla}^s \text{Rm}; \text{IV})} \\
&+ \underbrace{\sum_{\substack{p \geq 0, q \geq 1 \\ p+q=k+1 \\ i=0,1,\dots,r}} \nabla_\phi^i \bar{\nabla}_\phi^p \text{Rm}_\phi * \nabla_\phi^{r-i} \bar{\nabla}_\phi^q \text{Rm}_\phi}_{(\nabla^r \bar{\nabla}^s \text{Rm}; \text{V})}.
\end{aligned}$$

By (4.13), (4.14), (4.15), (4.16) and the uniform bound of $|D^{k+1}U|_{\omega_\phi}^2$, we can estimate the first term as follows:

$$\begin{aligned}
|(\nabla^r \bar{\nabla}^s \text{Rm}; \text{I})|_{\omega_\phi} &= |\nabla_\phi^r \bar{\nabla}_\phi^s (\text{Rm}_\phi * \text{Rm}_\phi + \bar{\nabla}_\phi \nabla_\phi \tilde{\eta} + \text{Rm}_\phi + \tilde{\eta} * \text{Rm}_\phi + \bar{\nabla}_\phi \nabla_\phi^2 X + \nabla_\phi X * \text{Rm}_\phi)|_{\omega_\phi} \\
&\leq C_{54} (1 + |\nabla_\phi^r \bar{\nabla}_\phi^s (\text{Rm}_\phi * \text{Rm}_\phi)|_{\omega_\phi} + |\nabla_\phi^r \bar{\nabla}_\phi^{s+1} \nabla_\phi \tilde{\eta}|_{\omega_\phi} + |\nabla_\phi^r \bar{\nabla}_\phi^{s+1} \nabla_\phi^2 X|_{\omega_\phi}), \\
|\nabla_\phi^r \bar{\nabla}_\phi^s (\text{Rm}_\phi * \text{Rm}_\phi)|_{\omega_\phi} + |\nabla_\phi^r \bar{\nabla}_\phi^{s+1} \nabla_\phi \tilde{\eta}|_{\omega_\phi} &\leq C_{55} (1 + |\nabla_\phi^r \bar{\nabla}_\phi^s \text{Rm}_\phi|_{\omega_\phi}), \\
|\nabla_\phi^r \bar{\nabla}_\phi^{s+1} \nabla_\phi^2 X|_{\omega_\phi} &\leq C_{56} (1 + |\nabla_\phi^r \bar{\nabla}_\phi^s \text{Rm}_\phi|_{\omega_\phi} + |\nabla_\phi^r \bar{\nabla}_\phi^s \nabla_\phi \text{Rm}_\phi|_{\omega_\phi}) \\
&\leq C_{57} (1 + |\nabla_\phi^r \bar{\nabla}_\phi^s \text{Rm}_\phi|_{\omega_\phi} + |\nabla_\phi^{r+1} \bar{\nabla}_\phi^s \text{Rm}_\phi|_{\omega_\phi}) \\
&\quad (\text{where we used the Ricci identity and } (H_k)).
\end{aligned}$$

Other terms are easier and estimated as follows:

$$|(\nabla^r \bar{\nabla}^s \text{Rm}; \text{II})|_{\omega_\phi} + |(\nabla^r \bar{\nabla}^s \text{Rm}; \text{III})|_{\omega_\phi} \leq C_{58},$$

$$|(\nabla^r \bar{\nabla}^s \text{Rm}; \text{IV})|_{\omega_\phi} + |(\nabla^r \bar{\nabla}^s \text{Rm}; \text{V})|_{\omega_\phi} \leq C_{59} (1 + |\nabla_\phi^r \bar{\nabla}_\phi^s \text{Rm}_\phi|_{\omega_\phi}).$$

Hence we have

$$\begin{aligned}
(4.26) \quad \left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) |\nabla_\phi^r \bar{\nabla}_\phi^s \text{Rm}_\phi|_{\omega_\phi}^2 &\leq C_{60} |\nabla_\phi^r \bar{\nabla}_\phi^s \text{Rm}_\phi|_{\omega_\phi}^2 - \frac{1}{2} |\nabla_\phi^{r+1} \bar{\nabla}_\phi^s \text{Rm}_\phi|_{\omega_\phi}^2 - |\bar{\nabla}_\phi \nabla_\phi^r \bar{\nabla}_\phi^s \text{Rm}_\phi|_{\omega_\phi}^2.
\end{aligned}$$

We can estimate the evolution equation of $|\nabla_\phi^{r-1}\bar{\nabla}_\phi^s\text{Rm}_\phi|_{\omega_\phi}^2$ in a similar way to get (4.27)

$$\left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) |\nabla_\phi^{r-1}\bar{\nabla}_\phi^s\text{Rm}_\phi|_{\omega_\phi}^2 \leq C_{61} - \frac{1}{2} |\nabla_\phi^r\bar{\nabla}_\phi^s\text{Rm}_\phi|_{\omega_\phi}^2 - |\bar{\nabla}_\phi\nabla_\phi^{r-1}\bar{\nabla}_\phi^s\text{Rm}_\phi|_{\omega_\phi}^2.$$

We take a smooth cut-off function τ that is identically equal to 1 on $\overline{B_{r'_{k+3}}(p)}$, vanishes on the outside of $B_{r'_{k+3}}(p)$ and satisfies

$$|\partial\tau|_{\omega_0}, |\sqrt{-1}\partial\bar{\partial}\tau|_{\omega_0} \leq C_{62},$$

where $r'_{k+3} > r''_{k+3} > r/2$. Applying the maximum principle to the function $\tau^2|\nabla_\phi^r\bar{\nabla}_\phi^s\text{Rm}_\phi|_{\omega_\phi}^2 + A_1|\nabla_\phi^{r-1}\bar{\nabla}_\phi^s\text{Rm}_\phi|_{\omega_\phi}^2$ (for a suitable uniform constant A_1), we get

$$|\nabla_\phi^r\bar{\nabla}_\phi^s\text{Rm}_\phi|_{\omega_\phi}^2 \leq C_{63}$$

on $\overline{B_{r''_{k+3}}(p)} \times [0, T]$.

Case 2: $s = 0$.

Using the Ricci identity repeatedly, we have

$$\begin{aligned} \left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) \nabla_\phi^{k+1}\text{Rm}_\phi &= \underbrace{\nabla_\phi^{k+1}\left(\frac{d}{dt} - \Delta_{\omega_\phi}\right)\text{Rm}_\phi}_{(\nabla^{k+1}\text{Rm}_\phi;\text{I})} + \underbrace{\sum_{\substack{p \geq 0, q \geq 1 \\ p+q=k+1}} \nabla_\phi^p\text{Rm}_\phi * \nabla_\phi^q(\text{Rm}_\phi + \tilde{\eta} + \nabla_\phi X)}_{(\nabla^{k+1}\text{Rm}_\phi;\text{II})} \\ &+ \underbrace{\sum_{\substack{p \geq 0, q \geq 1 \\ p+q=k+1}} \nabla_\phi^p\text{Rm}_\phi * \nabla_\phi^q\text{Rm}_\phi}_{(\nabla^{k+1}\text{Rm}_\phi;\text{III})}. \end{aligned}$$

By (4.13), (4.14), (4.15), (4.16) and the uniform bound of $|D^{k+1}U|_{\omega_\phi}^2$, we can estimate these terms as

$$|(\nabla^{k+1}\text{Rm}_\phi;\text{I})|_{\omega_\phi} \leq C_{64}(1 + |\nabla_\phi^{k+1}\text{Rm}_\phi|_{\omega_\phi} + |\nabla_\phi^{k+2}\text{Rm}_\phi|_{\omega_\phi}),$$

$$|(\nabla^{k+1}\text{Rm}_\phi;\text{II})|_{\omega_\phi} + |(\nabla^{k+1}\text{Rm}_\phi;\text{III})|_{\omega_\phi} \leq C_{65}(1 + |\nabla_\phi^{k+1}\text{Rm}_\phi|_{\omega_\phi}).$$

Thus we have

$$(4.28) \quad \left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) |\nabla_\phi^{k+1}\text{Rm}_\phi|_{\omega_\phi}^2 \leq C_{66}|\nabla_\phi^{k+1}\text{Rm}_\phi|_{\omega_\phi}^2 - \frac{1}{2}|\nabla_\phi^{k+2}\text{Rm}_\phi|_{\omega_\phi}^2 - |\bar{\nabla}_\phi\nabla_\phi^{k+1}\text{Rm}_\phi|_{\omega_\phi}^2.$$

Now we use the same cut-off function τ constructed in Case 1, and consider the function $\tau^2|\nabla_\phi^{k+1}\text{Rm}_\phi|_{\omega_\phi}^2 + A_2|\nabla_\phi^k\text{Rm}_\phi|_{\omega_\phi}^2$ (for a suitable uniform constant A_2). Since the evolution equation of $|\nabla_\phi^k\text{Rm}_\phi|_{\omega_\phi}^2$ has been already estimated in (4.25), the maximum principle implies that

$$|\nabla_\phi^{k+1}\text{Rm}_\phi|_{\omega_\phi}^2 \leq C_{67}$$

on $\overline{B_{r''_{k+3}}(p)} \times [0, T]$. Combining with Case 1, we have

$$|D^{k+1}\text{Rm}_\phi|_{\omega_\phi}^2 \leq C_{68}$$

on $\overline{B_{r''_{k+3}}(p)} \times [0, T]$, where the constant C_{68} depends only on $N, \gamma, \omega_0, X, \|\phi(\cdot, 0)\|_{C^{k+5}(B_r(p))}, \|\phi\|_{C^0(B_r(p) \times [0, T])}, \|\tilde{\eta}\|_{C^{k+3}(B_r(p))}$ and $\|F\|_{C^0(B_r(p))}$.

Applying D^{k+1} to the equation (4.2) and taking the trace, we have

$$\begin{aligned} |\Delta_{\omega_\phi} D^{k+1} \dot{\phi}|_{\omega_\phi} &\leq |D^{k+1} \Delta_{\omega_\phi} \dot{\phi}|_{\omega_\phi} + C_{69} \sum_{i=0}^{k+1} |D^i \text{Rm}_\phi|_{\omega_\phi} |D^{k+1-i} \dot{\phi}|_{\omega_\phi} \\ &\leq C_{70} \left(|D^{k+1} \text{Rm}_\phi|_{\omega_\phi} + |D^{k+1} \tilde{\eta}|_{\omega_\phi} + |D^{k+2} X|_{\omega_\phi} + \sum_{i=0}^{k+1} |D^i \text{Rm}_\phi|_{\omega_\phi} |D^{k+1-i} \dot{\phi}|_{\omega_\phi} \right). \end{aligned}$$

From the above estimates and (H_k) , we know that $|\Delta_{\omega_\phi} D^{k+1} \dot{\phi}|_{\omega_\phi}$ is uniformly bounded. Hence $D^{k+1} \dot{\phi}$ is $C^{1,\alpha}$, which implies $\dot{\phi}$ is $C^{k+2,\alpha}$. Differentiating the equation (4.3) $(k+2)$ -times and applying the elliptic Schauder estimates, we find that ϕ is $C^{k+4,\alpha}$ on $\overline{B_{r''_{k+3}}(p)} \times [0, T]$ where $r''_{k+3} > r_{k+4} > r/2$. Thus we have the statement (H_{k+1}) as desired. This completes the proof of Proposition 4.1. \square

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $T > 0$ be a constant. By Proposition 3.1, we know that

$$\sup_{M \times [0, T]} |\varphi_\epsilon|, \quad \sup_{M \times [0, T]} |\dot{\varphi}_\epsilon| < C(T)$$

for some constant $C(T)$ (independent of ϵ). Thus Proposition 3.2 implies that

$$(4.29) \quad A(T)^{-1} \omega_\epsilon \leq \omega_{\varphi_\epsilon} \leq A(T) \omega_\epsilon$$

on M for some constant $A(T)$ (independent of ϵ). We exhaust $M \setminus D$ by a sequence of compact subsets K , and $[0, \infty)$ by a sequence of closed intervals $[0, T]$. From (4.29), we know that

$$N^{-1} \omega_0 \leq \omega_{\phi_\epsilon} \leq N \omega_0$$

on $K \times [0, T]$, where the constant N only depends on K and T . Moreover, the initial data $k\chi + c_{\epsilon 0}, (1 - \beta)\eta_\epsilon, F_\epsilon$ are uniformly bounded in the C_{loc}^∞ -topology on $K \times [0, T]$. Thus Proposition 4.1, together with the diagonal argument implies that there exists a subsequence $\varphi_{\epsilon_i}(t)$ which converges to a function $\varphi(t)$ that is smooth on $M \setminus D$. Then, by (4.29), we also know that ω_φ is a conical Kähler metric along $(1 - \beta)D$. Now we will check that ω_φ satisfies the equation (1.2). Let $\zeta = \zeta(x, t)$ be any smooth $(n-1, n-1)$ -form on $M \times [0, \infty)$ with compact support $\text{Supp}(\zeta)$. Without loss of generality, we assume that $\text{Supp}(\zeta) \subset [0, T]$. Since $F_\epsilon, \chi, \varphi_\epsilon$ are uniformly bounded on $M \times [0, T]$, for $t \in [0, T]$, dominated convergence theorem

implies that

$$\begin{aligned}
\int_M \frac{\partial \omega_{\varphi_\epsilon}}{\partial t} \wedge \zeta &= \int_M \sqrt{-1} \partial \bar{\partial} \left(\log \left(\frac{\omega_{\varphi_\epsilon}^n}{\omega_0^n} \cdot \prod_{i=1}^d (\epsilon^2 + |s_i|_{H_i}^2)^{(1-\beta)\tau_i} \right) + F_0 + \gamma(k\chi + \varphi_\epsilon) \right) \wedge \zeta \\
&\quad + \int_M L_X \omega_{\varphi_\epsilon} \wedge \zeta \\
&= \int_M \left(\log \left(\frac{\omega_{\varphi_\epsilon}^n}{\omega_0^n} \cdot \prod_{i=1}^d (\epsilon^2 + |s_i|_{H_i}^2)^{(1-\beta)\tau_i} \right) + F_0 + \gamma(k\chi + \varphi_\epsilon) \right) \wedge \sqrt{-1} \partial \bar{\partial} \zeta \\
&\quad - \int_M \omega_{\varphi_\epsilon} \wedge L_X \zeta \\
&\xrightarrow{\epsilon_i \rightarrow 0} \int_M \left(\log \frac{\omega_\varphi^n}{\omega_0^n} + F_0 + \gamma(k\chi + \varphi) + \log |s_D|_{H_D}^{2(1-\beta)} \right) \wedge \sqrt{-1} \partial \bar{\partial} \zeta \\
&\quad - \int_M \omega_\varphi \wedge L_X \zeta \\
&= \int_M \sqrt{-1} \partial \bar{\partial} \left(\log \frac{\omega_\varphi^n}{\omega_0^n} + F_0 + \gamma(k\chi + \varphi) + \log |s_D|_{H_D}^{2(1-\beta)} \right) \wedge \zeta \\
&\quad + \int_M L_X \omega_\varphi \wedge \zeta \\
&= \int_M (-\text{Ric}(\omega_\varphi) + \gamma \omega_\varphi + (1-\beta)[D] + L_X \omega_\varphi) \wedge \zeta, \\
&\quad \int_M \omega_{\varphi_{\epsilon_i}} \wedge \frac{\partial \zeta}{\partial t} \xrightarrow{\epsilon_i \rightarrow 0} \int_M \omega_\varphi \wedge \frac{\partial \zeta}{\partial t}.
\end{aligned}$$

On the other hand, as in the proof of [LZ17, Theorem 4.1], we have

$$\int_M \frac{\partial \omega_{\varphi_\epsilon}}{\partial t} \wedge \zeta \xrightarrow{\epsilon_i \rightarrow 0} \int_M \frac{\partial \omega_\varphi}{\partial t} \wedge \zeta.$$

Hence, on $[0, T]$, we find that

$$\begin{aligned}
\frac{\partial}{\partial t} \int_M \omega_\varphi \wedge \zeta &= \int_M (-\text{Ric}(\omega_\varphi) + \gamma \omega_\varphi + (1-\beta)[D] + L_X \omega_\varphi) \wedge \zeta \\
&\quad + \int_M \omega_\varphi \wedge \frac{\partial \zeta}{\partial t}.
\end{aligned}$$

Integrating the above equation on $[0, \infty)$, we get

$$\begin{aligned}
\int_{M \times [0, \infty)} \frac{\partial \omega_\varphi}{\partial t} \wedge \zeta dt &= \int_0^\infty \left(\frac{\partial}{\partial t} \int_M \omega_\varphi \wedge \zeta - \int_M \omega_\varphi \wedge \frac{\partial \zeta}{\partial t} \right) dt \\
&= \int_{M \times [0, \infty)} (-\text{Ric}(\omega_\varphi) + \gamma \omega_\varphi + (1-\beta)[D] + L_X \omega_\varphi) \wedge \zeta dt.
\end{aligned}$$

Since ζ is arbitrary, ω_φ satisfies the equation (1.2) in the sense of distributions on $M \times [0, \infty)$. Meanwhile, the equation (2.6) can be written as

$$\frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \phi_\epsilon)^n}{\omega_0^n} = \frac{\exp(\dot{\phi}_\epsilon - F_0 - \gamma \phi_\epsilon - \theta_X - X(\phi_\epsilon))}{\prod_{i=1}^d (\epsilon^2 + |s_i|_{H_i}^2)^{(1-\beta)\tau_i}},$$

where ϕ_ϵ , $\dot{\phi}_\epsilon$ and $X(\phi_\epsilon)$ are uniformly bounded, which implies that the L^p -norm of the RHS is uniformly bounded for some $p > 1$ since $\beta \in (0, 1]$. Thus the Hölder continuity of φ with respect to ω_0 is a direct consequence from Kolodziej's work [Kol08, Theorem 2.1]. This completes the proof of Theorem 1.1. \square

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MATHEMATICAL INSTITUTE, TOHOKU UNIVERSITY, 6-3, AOBA, ARAMAKI, AOBA-KU, SENDAI, 980-8578, JAPAN

E-mail address: ryosuke.takahashi.a7@tohoku.ac.jp